UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in: MAT3360 — Introduction to partial differential equations

Day of examination: Wednesday, August 18, 2021

Examination hours: 09:00 – 13:00

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 10%)

Consider the PDE

$$\begin{cases} u_t + \tan(x) u_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \sin(x). \end{cases}$$

Find a solution to this initial value problem.

Løsningsforslag: The characteristic equation is

$$x' = \tan(x), \quad x(0) = x_0,$$

with solution

$$x_0 = \sin^{-1}\left(e^{-t}\sin(x)\right).$$

Hence a solution of the PDE is

$$u(x,t) = u(x_0, 0) = e^{-t}\sin(x).$$

Problem 2

Consider the function $f : [0,1] \to \mathbb{R}$ defined by yes

$$f(x) = \begin{cases} x & x \in [0, 1/2), \\ 1 - x & x \in [1/2, 1]. \end{cases}$$

2a (weight 10%)

Find the Fourier sine series of f.

(Continued on page 2.)

Løsningsforslag: For $k = 1, 2, 3, \dots$

$$\begin{aligned}
b_k &= 2\int_0^{1/2} x\sin(k\pi x)\,dx + 2\int_{1/2}^1 (1-x)\sin(k\pi x)\,dx \\
&= -2x\frac{1}{k\pi}\cos(k\pi x)\Big|_0^{1/2} + \frac{2}{k\pi}\int_0^{1/2}\cos(k\pi x)\,dx \\
&\quad - 2(1-x)\frac{1}{k\pi}\cos(k\pi x)\Big|_{1/2}^{1} - \frac{2}{k\pi}\int_{1/2}^1 \cos(k\pi x)\,dx \\
&= -\frac{1}{k\pi}\cos(k\frac{\pi}{2}) + \frac{2}{(k\pi)^2}\sin(k\frac{\pi}{2}) \\
&\quad + \frac{1}{k\pi}\cos(k\frac{\pi}{2}) + \frac{2}{(k\pi)^2}\sin(k\frac{\pi}{2}) \\
&= \frac{4}{(k\pi)^2}\sin(k\frac{\pi}{2}) \\
&= \frac{4}{(k\pi)^2}\begin{cases} 0 & k \text{ even} \\ (-1)^{\frac{k-1}{2}} & k \text{ odd.} \end{cases}
\end{aligned}$$

Hence the sine series reads

$$f(x) \sim \sum_k b_k \sin(k\pi x) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin((2k+1)\pi x).$$

2b (weight 10%)

Use the above sine series to calculate

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}.$$

Løsningsforslag: The function f is continuous on $[0, 1]$, hence Parsival's equality holds

$$\frac{1}{12} = \int_0^1 f^2(x)\,dx = \frac{16}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\frac{1}{2}.$$

Rearranging we get

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{96} \approx 1.02.$$

2c (weight 10%)

Explain why the Fourier sine series of f converges uniformly to a function g for all $x \in \mathbb{R}$ and sketch the graph of g for $x \in [-2, 2]$.

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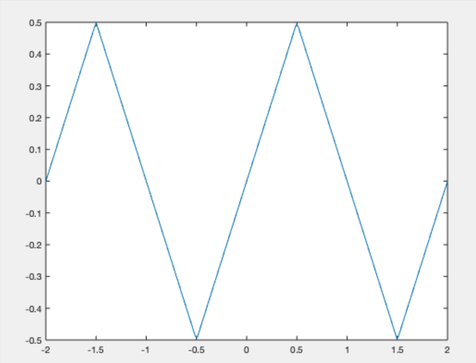
Løsningsforslag: The odd extension of f is continuous and its derivative is piecewise continuous, therefore the sine series converges uniformly to the odd periodic extension of f. The graph of g looks like this:



Problem 3

Consider the wave equation

$$\begin{cases} u_{tt} - u_{xx} = -u, \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0, \\ u(x,0) = f(x), \ u_t(x,0) = g(x), \ x \in (0,1), \end{cases} \tag{1}$$

where f and g are given smooth functions.

3a (weight 10%)

Find a formal solution of (1).

Løsningsforslag: We write $u = XT$, where $X \in C_0^2((0,1))$ and find that

$$\frac{T''}{T} = \frac{X''}{X} - 1 = \lambda.$$

Therefore

$$X'' = (\lambda + 1)X$$

which implies that $\lambda + 1 = -(k\pi)^2$ and $X = X_k(x) = \sin(k\pi x)$ for $k = 1, 2, 3, \ldots$. This then implies that

$$T = T_k(t) = a_k \cos(\sqrt{k^2\pi^2 + 1}t) + b_k \sin(\sqrt{k^2\pi^2 + 1}t)$$

where a_k and b_k are constants. Since the equation is linear (and we are formal) infinite linear combinations of solutions are solutions, and we have

$$u(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos(\sqrt{k^2\pi^2 + 1}t) + b_k \sin(\sqrt{k^2\pi^2 + 1}t) \right) \sin(k\pi x).$$

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We find the constants from the Fourier expansions of f and g, viz.

$$a_k = 2\int_0^1 f(x)\sin(k\pi x)\,dx,$$
$$b_k = \frac{2}{\sqrt{k^2\pi^2+1}}\int_0^1 g(x)\sin(k\pi x)\,dx.$$

3b (weight 10%)

Let u be a smooth solution of (1) and define the "energy"

$$E(t) = \frac{1}{2}\int_0^1 u^2 + (u_t)^2 + (u_x)^2\ dx.$$

Prove that $E(t) = E(0)$.

Løsningsforslag: We differentiate

$$\begin{aligned} E'(t) &= \int_0^1 uu_t + u_t u_{tt} + u_x u_{tx}\,dx \\ &= \int_0^1 uu_t + u_t u_{tt} - u_{xx} u_t\,dx = 0, \end{aligned}$$

where we integrated by parts and used the equation.

3c (weight 10%)

Show that (1) can have at most one smooth solution.

Løsningsforslag: The difference between two solutions will satisfy (1) with $f = g = 0$, so that $E(t) = 0$ for all t. Hence the difference is zero.

Problem 4

Let B denote the unit disc in $\mathbb{R}^2$

$$B = \left\{(x,y)\ \mid\ x^2 + y^2 < 1\right\}$$

with boundary

$$\partial B = \left\{(x,y)\ \mid\ x^2 + y^2 = 1\right\}.$$

Consider the differential operator

$$L[u] = \Delta u + u_x + u_y,$$

acting on functions in $C^2(\bar{B})$ where $\bar{B}$ denotes the closure of B.

(Continued on page 5.)

4a (weight 10%)

Let v be a function in $C^2(\bar{B})$ such that

$$L[v](x,y) > 0 \text{ for } (x,y) \text{ in } B.$$

Show that

$$v(x,y) \leq \max_{(z,w)\in\partial B} v(z,w) \text{ for all } (x,y) \text{ in } B.$$

Løsningsforslag: Assume that we have a maximum at $(x_0,y_0) \in B$, then

$$\Delta v(x_0,y_0) \leq 0, \;\; v_x(x_0,y_0) = v_y(x_0,y_0) = 0.$$

We get the contradiction

$$0 < L[v](x_0,y_0) = \Delta v(x_0,y_0) + v_x(x_0,y_0) + v_y(x_0,y_0) \leq 0.$$

Hence the maxima of v on $\bar{B}$ must be on ∂B.

4b (weight 10%)

Define

$$h(x,y) = e^{x^2+y^2}.$$

Show that $L[h](x,y) > 0$ for $(x,y) \in B$.

Løsningsforslag:

$$\begin{aligned} L[h](x,y) &= h(x,y)(4 + 4(x^2+y^2)) + 2h(x,y)(x+y) \\ &= h(x,y)\left(2 + (1+x)^2 + (1+y)^2 + 3x^2 + 3y^2\right) > 2, \end{aligned}$$

for $(x,y) \in B$.

4c (weight 10%)

Let u be a smooth solution to the boundary value problem

$$\begin{cases} L[u](x,y) = 0, \;\; (x,y) \in B, \\ u(x,y) = g(x,y), \;\; (x,y) \in \partial B, \end{cases}$$

where g is a given twice continuously differentiable function. Show that

$$\min_{(z,w)\in\partial B} g(z,w) \leq u(x,y) \leq \max_{(z,w)\in\partial B} g(z,w)$$

for $(x,y) \in B$.

Løsningsforslag: Define $v_\varepsilon = u + \varepsilon h$ for $\varepsilon > 0$, then $L[v_\varepsilon] = L[u] + \varepsilon L[h] > 0$. By **a**), $U + \varepsilon h = v_\varepsilon \leq \max_{\partial B} g + \varepsilon e$. Letting $\varepsilon \to 0$ gives the right inequality. Similarly to **a**), we find that if $L[v] < 0$, $v(x,y) > \min_{\partial B} v$ for $(x,y) \in B$. Then we can define $v_\varepsilon = u - \varepsilon h$ and proceed analogously to prove the left inequality.

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