UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3360 — Introduction to partial differential equations

Day of examination: Wednesday 14 june 2023

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 10%)

Solve the following initial value problem

$$\begin{aligned} u_t(x,t) + \frac{x}{1+t^2} u_x(x,t) &= 1 \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= \phi(x) \qquad x \in \mathbb{R}, \end{aligned}$$

where $\phi \in C^1(\mathbb{R})$ is a given function.

Problem 2

We consider the boundary value problem

$$\left. \begin{aligned} -u''(x) + \alpha u(x) e^{-u^2(x)/2} &= f(x) \qquad x \in (0,1) \\ u(0) = u(1) &= 0 \end{aligned} \right\} \tag{1}$$

where we are given a constant $\alpha \geq 0$ and a function $f \in C^2([0,1])$, and we seek a solution $u \in C_0^2((0,1))$. Let us recall that $C_0^2((0,1)) := C^2((0,1)) \cap C([0,1])$.

2a (weight 10%)

Let L be the operator defined by

$$(Lu)(x) := -u''(x) + \alpha u(x) \exp(-u^2(x)/2). \tag{2}$$

Show that L is positive definite on $C_0^2((0,1))$ for any $\alpha \geq 0$.

2b (weight 10%)

Describe differential equation's order, if it is homogeneous or nonhomogeneous, and its type of boundary conditions. Motivate your answers.

Furthemore, use a mathematical argument to determine if the differential equation is linear or nonlinear for different values of $\alpha \geq 0$.

(Continued on page 2.)

2c (weight 10%)

Explain why the boundary value problem (1) with $\alpha = 0$ has at most one solution in $C_0^2((0,1))$.

2d (weight 10%)

Show that if $f(x) > \alpha \exp(-1/2)$ for all $x \in [0,1]$ and $u \in C_0^2((0,1))$ is a solution of (1), then the solution satisfies that

$$\min_{x\in[0,1]} u(x) = 0.$$

(Hint: Use that $x \exp^{-x^2/2} \leq e^{-1/2}$ for all $x \in \mathbb{R}$.)

2e (weight 10%)

We now consider (1) with $\alpha = 1$.

Describe a numerical method with uniform stepsize $h = \frac{1}{n+1}$ for solving the boundary value problem (1), such that your resulting system of equations can be written on the form

$$(L_h v)(x_i) = f(x_i) \qquad i = 1, 2, \ldots, n$$

for an operator L_h on the set of discrete functions $D_{h,0}$.

Let $f \in C^2([0,1])$ be such that there exists a unique solution $u \in C_0^2((0,1))$ to the boundary value problem (1). Define the truncation error τ_h for your numerical method and derive an upper bound for $\|\tau_h\|_{h,\infty} = \max_{i=1,\ldots,n} |\tau_h(x_i)|$. You may use that the fourth derivative of u satisfies

$$\|u^{(4)}\|_\infty \leq \|f''\|_\infty + 5\|f\|_\infty^2 + 1 =: C_f \tag{3}$$

(you do not need to prove (3)).

Problem 3

We consider the partial differential equation

$$\begin{aligned} u_t &= u_{xx} - 2u & x &\in (0,1), \quad t > 0 & (4)\\ u_x(0,t) &= 0, \quad u(1,t) = 0 & t &\geq 0 & (5)\\ u(x,0) &= f(x) & x &\in (0,1) & (6) \end{aligned}$$

where $f \in C([0,1])$ is a given function.

3a (weight 10%)

Show that the PDE (4)-(6) at most has one smooth solution.

(Hint: Consider the energy function $E(t) = \int_0^1 u^2(x,t)dx$.)

3b (weight 10%)

Compute a family of particular solutions to (4)-(5).

(Continued on page 3.)

3c (weight 10%)

Describe the formal solution to (4)-(6). If you have not computed the particular solutions in Problem 3b, then you can describe the formal solution given a family of particular solutions $u_k(x,t) = T_k(t)X_k(x)$ for $k = 0, 1, \ldots$.

Thereafter, determine the solution in the case when $f(x) = \cos(\pi x/2) - 3\cos(9\pi x/2)$.

Problem 4 (weight 10%)

What can you say about the regularity and the periodic properties of the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2\exp(\pi)(4k-1)^{11/3}} \cos((4k-1)\pi x) \quad ?$$

THE END