UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3360 — Introduction to partial differential equations

Day of examination: Wednesday 14 june 2023

Examination hours: 09:00 – 13:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 10%)

Solve the following initial value problem

$$u_t(x,t) + \frac{x}{1+t^2} u_x(x,t) = 1 \qquad x \in \mathbb{R}, \quad t > 0$$
$$u(x,0) = \phi(x) \qquad x \in \mathbb{R},$$

where $\phi \in C^1(\mathbb{R})$ is a given function.

Solution suggestion: The characteristic equation $x' = x/(1+t^2)$ has solution

$$x(t) = x_0 \exp(\arctan(t)) \quad \text{and} \quad x_0(x,t) = x \exp(-\arctan(t)) \, .$$

Integrating $\frac{d}{dt} u(x(t),t) = 1$, we obtain the solution

$$u(x,t) = u(x_0(x,t),0) + t = \phi(xe^{-\arctan(t)}) + t$$

Problem 2

We consider the boundary value problem

$$\left.\begin{aligned} -u''(x) + \alpha u(x) e^{-u^2(x)/2} &= f(x) \qquad x \in (0,1) \\ u(0) = u(1) &= 0 \end{aligned}\right\} \tag{1}$$

where we are given a constant $\alpha \geq 0$ and a function $f \in C^2([0,1])$, and we seek a solution $u \in C_0^2((0,1))$. Let us recall that $C_0^2((0,1)) := C^2((0,1)) \cap C([0,1])$.

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2a (weight 10%)

Let L be the operator defined by

$$(Lu)(x) := -u''(x) + \alpha u(x)\exp(-u^2(x)/2). \tag{2}$$

Show that L is positive definite on $C_0^2((0,1))$ for any $\alpha \geq 0$.

Solution suggestion: For every $u \in C_0^2((0,1))$, integration by parts yields

$$\begin{aligned}\langle Lu, u\rangle &= \int_0^1 -u''(x)u(x)\,dx + \alpha\int_0^1 u^2(x)\exp(-u^2(x)/2)\,dx \\ &= \Big[-u'u\Big]_0^1 + \int_0^1 (u'(x))^2\,dx + \alpha\int_0^1 u^2(x)\exp(-u^2(x)/2)\,dx \\ &\geq \int_0^1 (u'(x))^2\,dx \geq 0.\end{aligned}$$

Moreover,

$$\langle Lu, u\rangle = 0 \iff u' \equiv 0 \iff \underbrace{u(x) = u(0) + \int_0^x u'(s)ds = 0 \quad \forall x \in [0,1]}_{\text{meaning } u\equiv 0}$$

2b (weight 10%)

Describe differential equation's order, if it is homogeneous or nonhomogeneous, and its type of boundary conditions. Motivate your answers.

Furthemore, use a mathematical argument to determine if the differential equation is linear or nonlinear for different values of $\alpha \geq 0$.

Solution suggestion: The equation is second order, since the highest order of derivative is second order. It is non-homogeneous whenever its righthandside $f(x)$ not is the zero-function. And the boundary conditions are Dirichlet, since they describe the values of the solution itself on the boundary.
And for $\alpha = 0$, the operator $Lu = -u''$ is linear, since for all $\alpha, \beta \in \mathbb{R}$ and $u, v \in C^2(0,1)$,

$$L(\alpha u + \beta v) = -(\alpha u + \beta v)'' = \alpha(-u'') + \beta(-v'') = \alpha L(u) + \beta L(v).$$

For $\alpha > 0$, the operator is nonlinear, as for the function $u, v = 1$ which belongs to $C^2(0,1)$, we have that

$$L(u+v) = -(u+v)''+\alpha(u+v)\exp(-(u+v)^2/2) = 2\alpha\exp(-2) \neq L(u)+L(v),$$

since

$$L(u)+L(v) = -u''+\alpha u\exp(-u^2/2)+(-v'')+\alpha v\exp(-v^2/2) = 2\alpha\exp(-\frac{1}{2}).$$

(Continued on page 3.)

2c (weight 10%)

Explain why the boundary value problem (1) with $\alpha = 0$ has at most one solution in $C_0^2((0,1))$.

Solution suggestion: When $\alpha = 0$, assume that $u, v \in C_0^2((0,1))$ both are solutions of the equation. Then, since L is linear, the function $(u - v) \in C_0^2((0,1))$ solves the homogeneous boundary value problem $L(u - v) = 0$ with $(u - v)(0) = (u - v)(1) = 0$. By the positive definiteness of L on C_0^2 we obtain that

$$\langle L(u - v), u - v \rangle = 0 \implies u \equiv v,$$

hence the solution is unique.

2d (weight 10%)

Show that if $f(x) > \alpha \exp(-1/2)$ for all $x \in [0,1]$ and $u \in C_0^2((0,1))$ is a solution of (1), then the solution satisfies that

$$\min_{x \in [0,1]} u(x) = 0.$$

(Hint: Use that $x \exp^{-x^2/2} \leq e^{-1/2}$ for all $x \in \mathbb{R}$.)

Solution suggestion: Using the hint, we have that

$$-u''(x) = f(x) - \alpha u(x) e^{-u^2(x)/2} \geq f(x) - \alpha e^{-1/2} > 0 \quad \forall x \in (0,1).$$

This means that u is strictly superharmonic on $(0,1)$: $u'' < 0$, and the result follows by a maximum principle: Suppose u has a local minimum in an interior point $x_0 \in (0,1)$. Then it must hold that $u''(x_0) \geq 0$, which is contradicted by $u'' < 0$. So u has no interior minima and

$$\min_{x \in [0,1]} u(x) = \min(u(0), u(1)) = 0.$$

2e (weight 10%)

We now consider (1) with $\alpha = 1$.

Describe a numerical method with uniform stepsize $h = \frac{1}{n+1}$ for solving the boundary value problem (1), such that your resulting system of equations can be written on the form

$$(L_h v)(x_i) = f(x_i) \qquad i = 1, 2, \ldots, n$$

for an operator L_h on the set of discrete functions $D_{h,0}$.

Let $f \in C^2([0,1])$ be such that there exists a unique solution $u \in C_0^2((0,1))$ to the boundary value problem (1). Define the truncation error

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τ_h for your numerical method and derive an upper bound for $\|\tau_h\|_{h,\infty} = \max_{i=1,\dots,n}|\tau_h(x_i)|$. You may use that the fourth derivative of u satisfies

$$\|u^{(4)}\|_\infty \le \|f''\|_\infty + 5\|f\|_\infty^2 + 1 =: C_f \tag{3}$$

(you do not need to prove (3)).

Solution suggestion: Let $h = 1/(n+1)$, $x_i = ih$ and $v_i = v(x_i) \approx u(x_i)$ for $i = 0, 1, \dots, n+1$ be the solution of the following system of equations: $v_0 = v_{n+1} = 0$ and

$$\underbrace{\frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2} + v_i e^{-v_i^2/2}}_{=(L_h v)(x_i)} = f(x_i) \qquad i = 1, 2, \dots, n.$$

Then the truncation error is the vector given by

$$\tau_h(x_i) = (L_h u)(x_i) - f(x_i) = \frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} + u(x_i)e^{-u^2(x_i)/2} - f(x_i)$$

Taylor expansion of u around the point x_i yields

$$\frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} = -u''(x_i) - \frac{u^{(4)}(\alpha) + u^{(4)}(\beta)}{24}h^2$$

for some $\alpha, \beta \in [x_{i-1}, x_{i+1}]$. We conclude that

$$|\tau_h(x_i)| \le \underbrace{|-u''(x_i) + u(x_i)e^{-u^2(x_i)/2} - f(x_i)|}_{=0} + \frac{1}{12}\|u^{(4)}\|_\infty h^2 \le C_f \frac{h^2}{12}$$

for all $i = 1, 2, \dots, n$, which means that $\|\tau_h\|_{h,\infty} \le C_f h^2/12$.

Problem 3

We consider the partial differential equation

$$u_t = u_{xx} - 2u \qquad x \in (0,1), \quad t > 0 \tag{4}$$

$$u_x(0,t) = 0, \quad u(1,t) = 0 \qquad t \ge 0 \tag{5}$$

$$u(x,0) = f(x) \qquad x \in (0,1) \tag{6}$$

where $f \in C([0,1])$ is a given function.

3a (weight 10%)

Show that the PDE (4)-(6) at most has one smooth solution.

(Hint: Consider the energy function $E(t) = \int_0^1 u^2(x,t)dx$.)

Solution suggestion: For $t > 0$, integration by parts yields

$$E'(t) = 2\int_0^1 u_t u\,dx = 2\int_0^1 (u_{xx} - 2u)u\,dx = \int_0^1 -2u_x^2 - 4u^2\,dx \le 0.$$

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Hence

$$\int_0^1 (u(x,t))^2 dx = E(t) \leq E(0) = \int_0^1 f(x)^2\, dx \qquad t \geq 0,$$

holds for any smooth solution of the PDE.
If u and v are smooth solutions of the PDE(4)-(6). Since $u(\cdot,0) = f$ and $v(\cdot,0) = f$, since the PDE is linear, and since the linear boundary conditions carry over to $u - v$ (meaning $(u-v)_x(0,t) = 0$ and $(u-v)(1,t) = 0$ for $t \geq 0$), the function $(u - v)$ solves (4)-(5). The energy argument therefore applies to $u - v$, so that for all $t \geq 0$,

$$\int_0^1 (u(x,t) - v(x,t))^2\, dx \leq \int_0^1 (u(x,0) - v(x,0))^2\, dx = 0 \implies u \equiv v.$$

3b (weight 10%)

Compute a family of particular solutions to (4)-(5).

Solution suggestion: Solution ansatz $u(x,t) = T(t)X(x)$ and separation of variables leads to

$$\frac{T'}{T} = \frac{X'' - 2X}{X} = -\lambda$$

for some $\lambda \in \mathbb{R}$. For the eigenvalue problem

$$LX = -X''(x) + 2X(x) = \lambda X, \qquad x \in (0,1)$$

with boundary conditions $X'(0) = 0$ and $X(1) = 0$. Note that the positive definiteness of L for any such eigenpair (X, λ) we have that

$$\lambda\langle X, X\rangle = \langle LX, X\rangle = \int_0^1 (X')^2 + 2X^2\, dx = 2\langle X, X\rangle + \langle X', X'\rangle.$$

For eigenfunctions one always assume that $X \not\equiv 0$, and by $X(1) = 0$, this also implies that $X' \not\equiv 0$. We conclude from the above that

$$\lambda = \frac{2\langle X, X\rangle + \langle X', X'\rangle}{\langle X, X\rangle} > 2.$$

Writing $\beta = \sqrt{\lambda - 2}$, we can rewrite the above problem $X'' = -\beta^2 X$, with general solution

$$X(x) = a\cos(\beta x) + b\sin(\beta x).$$

$X'(0) = 0$ and $\beta > 0$ implies that $b = 0$, and $X(1) = 0$ implies that $\beta_k = (k + 1/2)\pi$ for $k = 0, 1, \ldots$. Eigenpairs:

$$X_k(x) = \cos((k{+}1/2)\pi x) \quad \text{with eigvals} \quad \lambda_k = \beta_k^2{+}2 = (k{+}1/2)^2\pi^2{+}2$$

and

$$T_k' = \lambda_k T \quad \text{has solution} \quad T_k(t) = \exp(-\lambda_k t).$$

Particular solutions

$$u_k(x,t) = T_k(t)X_k(x) = \exp\Big(-\big(2{+}(k{+}1/2)^2\pi^2\big)t\Big)\cos\big((k{+}1/2)\pi x\big)$$

for $k = 0, 1, \ldots$

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3c (weight 10%)

Describe the formal solution to (4)-(6). If you have not computed the particular solutions in Problem 3b, then you can describe the formal solution given a family of particular solutions $u_k(x,t) = T_k(t)X_k(x)$ for $k = 0, 1, \ldots$.

Thereafter, determine the solution in the case when $f(x) = \cos(\pi x/2) - 3\cos(9\pi x/2)$.

Solution suggestion:
Since the PDE is linear and the boundary conditions are preserved under linear combination of particular solutions, it holds that any linear combination of particular solutions also is a particular solution. The formal solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} c_k u_k(x,t) = \sum_{k=0}^{\infty} c_k T_k(t) X_k(x),$$

where

$$c_k = \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle} \ \frac{\int_0^1 f(x) X_k(x)\, dx}{\int_0^1 X_k^2(x)\, dx}.$$

When $f(x) = \cos(\pi x/2) - 3\cos(9\pi x/2)$, the unique solution is

$$\begin{aligned} u(x,t) &= u_0(x,t) - 3u_4(x,t) \\ &= \exp\Big(- \big(2 + (\pi/2)^2\big)t\Big) \cos\big(\pi x/2\big) \\ &\quad - 3\exp\Big(- \big(2 + (9\pi/2)^2\big)t\Big) \cos\big(9\pi x/2\big) \end{aligned}$$

Problem 4 (weight 10%)

What can you say about the regularity and the periodic properties of the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2\exp(\pi)(4k-1)^{11/3}} \cos((4k-1)\pi x) \quad ?$$

Solution suggestion: The Fourier coefficients are $a_{4k-1} = \frac{1}{2\exp(\pi)}(4k-1)^{-11/3}$. So for integers $m \in \mathbb{N}$, we have that

$$\sum_{k=1}^{\infty} k^{2m} a_k^2 = \frac{1}{4\exp(2\pi)} \sum_{k=1}^{\infty} k^{2m-22/3} = \begin{cases} < \infty & m \leq 3 \\ \infty & m \geq 4 \end{cases}$$

Since the above sum is bounded for $m = 3$ but not for $m \geq 4$, the theory on the decay of Fourier coefficients tells us that $f \in C^2([-1,1])$ and that it satisfies the following 2-periodic properties: $f^{(j)}(-1) = f^{(j)}(1)$ for all $0 \leq j \leq 2$.

THE END