UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3360 — Introduction to partial differential equations

Day of examination: Monday, June 10, 2024

Examination hours: 15:00 – 19:00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 20%)

For each of the four differential equations below, describe the following five properties:

1. is it an ordinary differential equation (ODE) or a partial differential equation (PDE)?
2. describe the differential operator of the homogeneous part of the differential equation (the domain and range of the operator mapping is not needed),
3. is the differential equation linear or nonlinear?
4. is the differential equation homogeneous or non-homogeneous?
5. what is the order of the differential equation?

Unlike for all later problems, you do not need to motivate your answers in Problem 1.

a)

$$u_t(x,t) = 2u_{xx}(x,t) + 3u_x(x,t) \qquad x \in \mathbb{R}, \quad t > 0\,.$$

b)

$$X''(x) = e^{X(x)} - \sin(3x^2) \qquad x \in \mathbb{R}\,.$$

c)

$$u_t(x,y,t) - xu_{xx}(x,y,t) - y^2 u_{yyy}(x,y,t) = xy^2t^3 \qquad (x,y) \in \mathbb{R}^2, \quad t > 0\,.$$

d)

$$u_t(x,t) + u_x(x,t)u(x,t) = 0 \qquad x \in \mathbb{R}, \quad t > 0\,.$$

(Continued on page 2.)

Solution suggestion:

a) 1. PDE. 2. $L(u) = u_t - 2u_{xx} - 3u_x$. 3. linear. 4. homogeneous. 5. order 2.

b) 1. ODE. 2. $L(X) = X'' - e^X$. 3. nonlinear. 4. non-homogeneous. 5. order 2.

c) 1. PDE. 2. $L(u)(x,y,t) = u_t(x,y,t) - xu_{xx}(x,y,t) - y^2u_{yyy}(x,y,t)$. 3. linear. 4. non-homogeneous. 5. order 3.

d) 1. PDE. 2. $L(u) = u_t + u_x u$. 3. nonlinear. 4. homogeneous. 5. order 1.

Problem 2

2a (weight 10%)

We consider the wave equation

$$\left.\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) && x \in (0,1), \quad t > 0 \\ u_x(0,t) &= 0, \quad u_x(1,t) = 0 && t \geq 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x) && x \in (0,1), \end{aligned}\right\} \tag{1}$$

where $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are given smooth functions. Show that the PDE (1) has at most one solution in $C^2([0,1] \times [0,\infty))$.

2b (weight 10%)

Find the unique solution of the wave equation (1) when $f(x) = 2\sin^2(\pi x)$ and $g(x) = \cos(3\pi x)$.

Hint: Rewrite $f(x)$ using trigonometric identities and determine the solution for instance through computing the formal solution to this problem.

Solution suggestion: a) Assume that $u, v \in C^2$ both solve the PDE both with the boundary conditions and initial conditions given in (1). Then, by the linearity of the PDE, $w := (u - v) \in C^2$ and it solves the wave equation $w_{tt} = w_{xx}$ with boundary conditions $w_x(0,t) = w_x(1,t) = 0$ for $t \geq 0$ and initial conditions $w(\cdot, 0) \equiv 0$ and $w_t(\cdot, 0) \equiv 0$.
Differentiating the energy function

$$E(t) = \int_0^1 (w_x(x,t))^2 + (w_t(x,t))^2 dx$$

and using that $w_{xt} = w_{tx}$, we obtain that

$$\begin{aligned} E'(t) &= 2\int_0^1 w_x w_{tx} + w_t w_{tt} dx \\ &= 2\Big((w_t w_x)(1,t) - (w_t w_x)(0,t)\Big) + 2\int_0^1 w_t(w_{tt} - w_{xx}) dx \\ &= 0. \end{aligned}$$

(Continued on page 3.)

This implies that $w_t \equiv 0$ and $w_x \equiv 0$, which means that $w \equiv c$ for some constant $c \in \mathbb{R}$, and $c = 0$ since $w(\cdot, 0) \equiv 0$. So $w \equiv 0$, which means that $u = v$, so the solution is unique (if it exists).

b) The formal solution of the wave equation is given by

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \cos(k\pi x)(a_k \cos(k\pi t) + \frac{b_k}{k\pi}\sin(k\pi t))$$

where

$$a_k = 2\int_0^1 \cos(k\pi x) f(x)dx, \quad \text{and} \quad b_k = 2\int_0^1 \cos(k\pi x) g(x)dx$$

By the orthogonality of the Cosine series on $[0,1]$ and using that

$$2\sin^2(\pi x) = -\cos(\pi x + \pi x) + \cos(\pi x - \pi x) = 1 - \cos(2\pi x),$$

we obtain that

$$a_k = \begin{cases} 2 & k = 0 \\ -1 & k = 2 \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad b_k = \begin{cases} 1 & k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

We obtain the formal solution

$$u(x,t) = 1 - \cos(2\pi x)\cos(2\pi t) + \frac{1}{3\pi}\cos(3\pi x)\sin(3\pi t).$$

Since the formal solution is a linear combination of a finite number of smooth particular solutions and the PDE is linear, it follows that this is indeed is the unique smooth solution of the PDE.

Problem 3

We consider the heat equation

$$\begin{aligned} u_t(x,t) &= 3u_{xx}(x,t) & & x \in (0,1), \quad t > 0 \\ u(0,t) &= 0, \quad u(1,t) = 0 & & t \geq 0 \\ u(x,0) &= x(1-x) & & x \in (0,1). \end{aligned}$$

3a (weight 10%)

Let $\Delta t > 0$ and let $\Delta x = 1/(n+1)$ for some integer $n \geq 1$. Construct an explicit finite difference method for numerically solving the heat equation on the mesh

$$(x_j, t_m) = (j\Delta x, m\Delta t) \qquad \text{for} \quad j = 0, 1, \ldots, n+1, \quad \text{and} \quad m \geq 0.$$

Remember to describe the boundary conditions and the initial condition.

Notation: For consistency with the problem formulation in Problem 3b, let v_j^m denote the numerical solution at the point (x_j, t_m).

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3b (weight 10%)

Impose a condition for the relationship between Δt and Δx such that

$$\max_{j\in\{0,1,\dots,n+1\}} v_j^m \leq 1/4, \qquad \text{holds for all} \quad m \geq 0, \tag{2}$$

where v_j^m denotes your numerical solution at the point (x_j, t_m) from Problem 3a. **Furthermore, verify that the inequality** (2) **holds under your imposed condition.**

Solution suggestion: a)We propose the scheme

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = 3\frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2}$$

for $j = 1, \dots, n$ and $m \geq 0$. Boundary conditions $v_0^m = v_{n+1}^m = 0$ for $m \geq 0$, and initial condition $v_j^0 = x_j(1 - x_j)$ for $j = 1, \dots, n$.

b) We impose that

$$\frac{6\Delta t}{\Delta x^2} \leq 1.$$

Let $V_+^m := \max_{j\in\{0,1,\dots,n+1\}} v_j^m$ and assume that (2) holds for some $m \geq 0$. The scheme tells us that

$$v_j^{m+1} = v_j^m \underbrace{(1 - \frac{6\Delta t}{\Delta x^2})}_{\geq 0} + \frac{3\Delta t}{\Delta x^2}\left(v_{j-1}^m + v_{j+1}^m\right) \qquad j = 1, \dots, n$$

so that

$$v_j^{m+1} = V_+^m(1 - \frac{3\Delta t}{2\Delta x^2}) + \frac{3\Delta t}{\Delta x^2}V_+^m = V_+^m$$

holds for all $j = 1, \dots, n$. Since also $v_0^{m+1} = v_{n+1}^{m+1} = 0 \leq 1/4$, we conclude that $V_+^m \leq 1/4 \implies V_+^{m+1}$. Observing that $V_+^0 \leq \max_{x\in[0,1]} x(1-x) = 1/4$, the result holds by induction.

Problem 4

Let $\Omega \subset \mathbb{R}^2$ be an open, non-empty, bounded, and connected domain with smooth boundary $\partial\Omega$, and let $n : \partial\Omega \to \mathbb{R}^2$ denote the unit outer normal vector. We consider the PDE

$$\left.\begin{aligned} -(e^{xy}u_x(x,y))_x - (e^{xy}u_y(x,y))_y &= f(x,y) \qquad (x,y) \in \Omega \\ \frac{\partial u}{\partial n}(x,y) + u(x,y) &= g(x,y) \qquad (x,y) \in \partial\Omega \end{aligned}\right\} \tag{3}$$

where $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are given smooth functions, and we recall the notation $\frac{\partial u}{\partial n} = u_x n_1 + u_y n_2$, where $n = (n_1, n_2)$.

4a (weight 10%)

Show that the differential operator $L : C^2(\Omega) \to C(\Omega)$ defined by $L(u)(x,y) = -(e^{xy}u_x(x,y))_x - (e^{xy}u_y(x,y))_y$ is linear.

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4b (weight 10%)

Show that the PDE (3) has at most one smooth solution.

Hint: Note that

$$L(u) = -\text{div}\begin{bmatrix} e^{xy}u_x \\ e^{xy}u_y \end{bmatrix}$$

and that by the boundedness of the domain Ω, it holds that $\min_{(x,y)\in\overline{\Omega}} e^{xy} =: c > 0$.

Solution suggestion: a) For any $u, v \in C^2$ and any $\alpha, \beta \in \mathbb{R}$, we have that

$$\begin{aligned} L(\alpha u + \beta v) &= -(e^{xy}(\alpha u_x + \beta v_x))_x - (e^{xy}(\alpha u_y + \beta u_y))_y \\ &= -\alpha\Big((e^{xy}u_x)_x + (e^{xy}u_y)_y\Big) - \beta\Big((e^{xy}v_x)_x + (e^{xy}v_y)_y\Big) \\ &= \alpha L(u) + \beta L(v). \end{aligned}$$

b)Assume that u and v both are smooth solutions of (3) with the same right-hand side f in the differential equation and $g(x)$ in the boundary condition.
Then by the linearity of the differential operator and the linearity of the homogeneous part of the boundary condition, $w = u - v$ is a smooth function that solves the PDE

$$\begin{aligned} L(w) = -(e^{xy}w_x(x,y))_x - (e^{xy}w_y(x,y))_y &= 0 \qquad (x,y) \in \Omega \\ \frac{\partial w}{\partial n}(x,y) + w(x,y) &= 0 \qquad (x,y) \in \partial\Omega. \end{aligned} \tag{4}$$

By the divergence theorem/Green's first identity,

$$\begin{aligned} 0 &= \iint_\Omega (wL(w))(x,y)\, dx\, dy \\ &= -\iint_\Omega w \text{ div}\begin{bmatrix} e^{xy}w_x \\ e^{xy}w_y \end{bmatrix} dx\, dy \\ &= \iint_\Omega e^{xy}|\nabla w|^2\, dx\, dy - \int_{\partial\Omega} e^{xy} w \frac{\partial w}{\partial n} ds \\ &= \iint_\Omega e^{xy}|\nabla w|^2\, dx\, dy + \int_{\partial\Omega} e^{xy} w^2 ds \\ &\geq c \iint_\Omega |\nabla w|^2\, dx\, dy + c\int_{\partial\Omega} w^2 ds\,. \end{aligned}$$

We obtained the fourth equation by using the boundary condition. This implies that $w_x \equiv 0$ and $w_y \equiv 0$, so that w is a constant function on Ω, and from the latter integral we see that $w|_{\partial\Omega} = 0$. Combined, this implies that $w \equiv 0$ on $\overline{\Omega}$, hence $u = v$.

(Continued on page 6.)

Problem 5

5a (weight 10%)

For $f(x) = x\exp(x^2)$ and $N \in \mathbb{N}$, let $S_N(f)$ denote the truncated Fourier series of f on $[-1,1]$, namely,

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x) + b_k \sin(k\pi x)\,,$$

where

$$a_k = \int_{-1}^{1} \cos(k\pi x) f(x)dx, \quad \text{and} \quad b_k = \int_{-1}^{1} \sin(k\pi x) f(x)dx.$$

Determine if it holds that $S_N(f)$ converges as $N \to \infty$ in

1. pointwise sense for all $x \in [-1,1]$, and if so, to what limit?
2. uniform sense on the interval $[-1,1]$ to f?

Hint: To answer case 2., it may be helpful to use that uniform convergence implies pointwise convergence to the same limit.

5b (weight 10%)

Show that

$$a_k = 0, \qquad \text{for all} \quad k = 0, 1, \ldots$$

and for the sequence

$$\lim_{k\to\infty} b_k = 0.$$

for the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ described in Problem 5a.

Solution suggestion: **a)** We have that $f \in C^1[-1,1]$ is on $[-1,1]$, but since f is not periodic, its 2-periodic extension from $[-1,1)$ has discontinuities only at odd integers $-1, 1, 3, \ldots$, as

$$f_{per}(-1-) = f(1-) = e^1 \quad \text{while} \quad f_{per}(-1+) = f(-1+) = -e^1.$$

The function f is continuously differentiable on $[-1,1]$, which implies that it is one-sided differentiable on $[-1,1]$. This implies pointwise convergence of $S_N(f)$, (as this implies that f_{per} is one-sided differentiable on $\mathbb{R}$, where we note that since f' is not 2-periodic, $f'(-1) \neq f'(1)$, $f'_{per}(\pm1-) \neq f'_{per}(\pm1+)$).
Since $f_{per} = f|_{(-1,1)}$ is continuous on $(-1,1)$, we have that $f_{per}(x+) = f_{per}(x-)$ for all $x \in (-1,1)$ and pointwise convergence holds with

$$\lim_{N\to\infty} S_N(f)(x) = \frac{f_{per}(x-) + f_{per}(x+)}{2} = \begin{cases} = (e^1 - e^1)/2 = 0 & x \in \{-1, 1\} \\ x\exp(x^2) & x \in (-1,1) \end{cases}.$$

Suppose next $S_N(f)(x)$ converges uniformly to f on $[-1,1]$. Then, by the pointwise convergence at $x = 1$, we reach the contradiction

$$0 = \lim_{N\to\infty} \|S_N(f) - f\|_\infty \leq \lim_{N\to\infty} |S_N(f)(1) - f(1)| = e^1.$$

(Continued on page 7.)

This shows that $S_N(f)$ does not converge uniformly to f on $[-1,1]$.

b) f is an odd function, as $f(-x) = -xe^{x^2} = -f(x)$. Therefore,

$$\begin{aligned}\int_{-1}^{1} \cos(k\pi x) f(x) dx &= \int_{-1}^{0} \cos(k\pi x) f(x) dx + \int_{0}^{1} \cos(k\pi x) f(x) dx \\ &= \int_{0}^{1} \cos(k\pi y) f(-y) dy + + \int_{0}^{1} \cos(k\pi x) f(x) dx \\ &= 0\end{aligned}$$

holds for all $k \geq 0$.

Since f is continuous on $[-1,1]$ and all $a_k = 0$, Bessel's inequality yields that

$$\sum_{k=1}^{\infty} b_k^2 \leq \int_0^1 f(x)^2 dx < \infty.$$

Hence

$$\lim_{N\to\infty} \sum_{k=N}^{\infty} b_k^2 = 0,$$

which implies that $b_k \to 0$ as $k \to \infty$.

THE END