Solutions to the exam in MAT3420, Spring 2021

Problem 1a.

Describe how one represents the pure states of a one-qubit system on the Bloch sphere. Draw a picture showing the images of

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \text{and} \qquad -|0\rangle.$$

Solution.
A pure state of a one-qubit system can be written as

$$e^{i\psi}\big(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\big),$$

with $0 \le \psi < 2\pi$, $0 \le \theta < \pi$, $0 \le \phi < 2\pi$. Its image on the sphere is then

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

For the pure state

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

we have $\psi = 0$, $\theta = \dfrac{\pi}{2}$, $\phi = 0$, so we get the point $(1, 0, 0)$ on the sphere, and for

$$-|0\rangle$$

we have $\psi = \pi$, $\theta = 0$ and ϕ any number in $[0, 2\pi)$, so we get the point $(0, 0, 1)$.

Problem 1b.

Prove the formula

$$e^{i\theta\vec{a}\cdot\vec{\sigma}} = (\cos\theta)I + i(\sin\theta)\vec{a}\cdot\vec{\sigma}$$

for all unit vectors $\vec{a} \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$.

Solution. For every unit vector $\vec{a} \in \mathbb{R}^3$, the operator $S = \vec{a}\cdot\vec{\sigma}$ is a nonscalar symmetry, that is, $S = S^*$, $S^2 = I$, $S \ne \pm I$. It follows that it is diagonalizable, with the eigenvalues ± 1. It suffices to check the formula on the eigenvectors of S. If $|\psi\rangle$ is an eigenvector of S with eigenvalue $\lambda \in \{-1, 1\}$, then

$$\begin{aligned} e^{i\theta S}|\psi\rangle &= e^{i\theta\lambda}|\psi\rangle = \cos(\lambda\theta)|\psi\rangle + i\sin(\lambda\theta)|\psi\rangle \\ &= (\cos\theta)|\psi\rangle + i\lambda(\sin\theta)|\psi\rangle = \big((\cos\theta)I + i(\sin\theta)S\big)|\psi\rangle. \end{aligned}$$

Problem 1c.

Describe in words the action of the unitaries $e^{i\theta\vec{a}\cdot\vec{\sigma}}$ on the pure states in the Bloch coordinates. You don't have to justify your answer.

Solution. For every unit vector $\vec{a} \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$, the unitary $e^{i\theta\vec{a}\cdot\vec{\sigma}}$ acts by the rotation by 2θ clockwise about the axis $\vec{a}$.

Problem 2a.

Give the definition of the Schmidt number of a pure state.

Solution. Every pure state of a composite system $H_1 \otimes H_2$ can be written as

$$\sum_{k=1}^{n} \lambda_k |\psi_k\rangle \otimes |\phi_k\rangle,$$

where $\lambda_k > 0$ and the systems $\{\psi_1, \dots, \psi_n\} \subset H_1$ and $\{\phi_1, \dots, \phi_n\} \subset H_2$ are orthonormal. The number n is called the Schmidt number of the state.

Problem 2b.

Compute the Schmidt number of
$$\frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

Solution. As we know, the Schmidt number of $|\omega\rangle \in H_1 \otimes H_2$ can be computed as follows. Choose a basis $|\psi_1\rangle, \dots, |\psi_m\rangle$ in H_1. Then
$$|\omega\rangle = \sum_{k=1}^{m} |\psi_k\rangle \otimes |\phi_k\rangle$$
for uniquely determined $|\phi_k\rangle \in H_2$, and then the Schmidt number of $|\omega\rangle$ is the dimension of the space spanned by $|\phi_1\rangle, \dots, |\phi_m\rangle$.

Since we can write
$$\frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = \frac{1}{2}|0\rangle \otimes (|0\rangle + |1\rangle) - \frac{1}{2}|1\rangle \otimes (|0\rangle - |1\rangle),$$
and the vectors
$$\frac{1}{2}(|0\rangle + |1\rangle) \qquad \text{and} \qquad -\frac{1}{2}(|0\rangle - |1\rangle)$$
are linearly independent, we conclude that the Schmidt number equals 2.

Problem 3a.

Find the continued fraction expansion of $\frac{23}{16}$.

Solution. We have
$$\frac{23}{16} = 1 + \left(\frac{16}{7}\right)^{-1}, \quad \frac{16}{7} = 2 + \left(\frac{7}{2}\right)^{-1}, \quad \frac{7}{2} = 3 + \frac{1}{2},$$
hence
$$\frac{23}{16} = [1; 2, 3, 2].$$

Problem 3b.

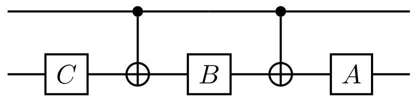
Find all rational numbers $\frac{p}{q}$ satisfying
$$\left|\frac{23}{16} - \frac{p}{q}\right| \leq \frac{1}{2q^2}.$$

Solution. We know that all such nonintegral numbers must be convergents of the continued fraction expansion of $\frac{23}{16}$. The convergents are
$$1, \quad [1; 2] = \frac{3}{2}, \quad [1; 2, 3] = \frac{10}{7}, \quad [1; 2, 3, 2] = \frac{23}{16}.$$
They all satisfy the required inequality. Also, the only integer satisfying the required inequality is 1. Therefore the complete list of required rational numbers is
$$1, \quad \frac{3}{2}, \quad \frac{10}{7}, \quad \frac{23}{16}.$$

Problem 4.

Assume A, B, C, U are one-qubit unitary gates satisfying $ABC = I$ and $AXBXC = U$, where $X = \sigma_x$ is the NOT gate. Consider the control-U gate $\Lambda(U)$, so
$$\Lambda(U)(|a\rangle \otimes |b\rangle) = |a\rangle \otimes U^a |b\rangle \quad \text{for all} \quad a, b \in \{0, 1\}.$$
Draw a quantum circuit expressing $\Lambda(U)$ in terms of A, B, C, X and the CNOT gates.

Solution.



Problem 5a.

Describe all separable pure states $|\phi\rangle \otimes |\psi\rangle$ of a 2-qubit system such that

$$\mathrm{CNOT}(|\phi\rangle \otimes |\psi\rangle)$$

is again separable, that is, it has the form $|\phi'\rangle \otimes |\psi'\rangle$.

Solution. If $|\phi\rangle = a|0\rangle + b|1\rangle$, then

$$\mathrm{CNOT}(|\phi\rangle \otimes |\psi\rangle) = a|0\rangle \otimes |\psi\rangle + b|1\rangle \otimes X|\psi\rangle.$$

We see that if $a = 0$ or $b = 0$, then the state we get is again separable. Assume now that $a, b \neq 0$. Then the Schmidt number of the above state is the dimension of the space spanned by $|\psi\rangle$ and $X|\psi\rangle$. The state is separable if and only if the Schmidt number equals 1, that is, $|\psi\rangle$ is an eigenvector of X. Therefore the required states have the form

$$|0\rangle \otimes |\psi\rangle, \quad |1\rangle \otimes |\psi\rangle, \quad |\phi\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |\phi\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

where $|\phi\rangle$ and $|\psi\rangle$ are arbitrary pure states.

Problem 5b.

Assume we are given a quantum circuit on k qubits, with input state $|0\dots0\rangle$, consisisting of n gates from our standard universal gate set $\{H, T^{\pm1}, \mathrm{CNOT}\}$ followed by a final measurement of all the qubits. Assume it is known that at every step of the computation the state we get is separable, that is, it is of the form

$$|\phi_1\rangle \otimes \dots \otimes |\phi_k\rangle.$$

Argue, without going into too many details, that such a quantum computation can be efficiently simulated on a classical computer. More precisely, show that, assuming we can do exact arithmetic operations with real numbers, we need not more than $C_k n$ such operations, for some constant C_k depending on k, to compute the probabilities of all possible outcomes $a_1 \dots a_k$ of the quantum computation.

Solution. By assumption at every step of the quantum computation we get a state $|\phi_1\rangle \otimes \dots \otimes |\phi_k\rangle$, which we can represent by the vector $(|\phi_1\rangle, \dots, |\phi_k\rangle) \in \mathbb{C}^{2k}$. What happens to this representation at the next step? If we act by a one-qubit gate A on the l-th qubit, we can simply apply A to $|\phi_l\rangle$. This requires a fixed small number of arithmetic operations (four multipilications and two additions of complex numbers). The action of CNOT is a bit more complicated: if we act on the qubits l and m, then our assumptions imply that $\mathrm{CNOT}(|\phi_l\rangle \otimes |\phi_m\rangle)$ is a simple tensor $|\phi'\rangle \otimes |\psi'\rangle$, but we need to find such a tensor explicitly. A simple algorithm for computing this tensor can be written down using the solution of Problem 5a, but even without that problem it is not difficult to see that this can be done using a finite number of arithmetic operations. It follows that a vector $(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle, \dots, \alpha_{k0}|0\rangle + \alpha_{k1}|1\rangle) \in \mathbb{C}^{2k}$ representing the final state

$$(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle) \otimes \dots \otimes (\alpha_{k0}|0\rangle + \alpha_{k1}|1\rangle)$$

of our quantum circuit can be computed by not more than Cn arithmetic operations for a universal constant C. The probability of every outcome $a_1 \dots a_k$ of our quantum computation equals

$$|\alpha_{1a_1} \dots \alpha_{ka_k}|^2,$$

so the number D_k of arithmetic operations needed to compute such probabilities from the final state depends only on k. We see that all in all we need not more than $Cn + D_k$ arithmetic operations to compute these probabilities, which is even better than what was claimed.