Superdense coding (lect. 9 in main ref.)

basic premise:

using an entangled state, we can send more information than the non-quantum schemes

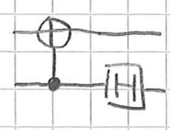
how it works

- Alice wants to send <u>2-bit information</u> to Bob
  $\hookrightarrow xy \quad x, y = 0, 1$.
- they start with a maximally entangled state
  $$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$
- Alice performs unitary operations on the first factor of $|\phi\rangle$ according to x, y
  $$|\psi\rangle = (U \otimes I_2)|\phi\rangle.$$
- Alice sends her qubit to Bob, so he has $|\psi\rangle$. (not just its second factor)
- Bob performs another op., $|\psi'\rangle = U'|\psi\rangle$ then measures both factors of $|\psi'\rangle$ to decide x and y

Alice's operation(s): $U = U_y U_x$.

if $x = 0$, $U_0 = I_2$, if $x = 1$ U_x is the Pauli-X gate $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

if $y = 0$, $U_1 = I_2$, if $y = 1$, U_y is the Pauli-Z gate $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Bob's operation $U' =$ 

i.e.

first apply the CNOT (but first & second factors reversed)

$$|*0\rangle \mapsto |*0\rangle, \quad |01\rangle \mapsto |11\rangle, \quad |11\rangle \mapsto |01\rangle$$

then the Hadamard gate $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ on the second factor why this works :

after Alice's op., the state $|\psi\rangle = |\psi_{xy}\rangle$ is one of :

$$|\psi_{00}\rangle = |\phi\rangle, \quad |\psi_{01}\rangle = (X \otimes I_2)|\phi\rangle = \tfrac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

$$|\psi_{10}\rangle = (Z \otimes I_2)|\phi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi_{11}\rangle = (ZX \otimes I_2)|\phi\rangle = \tfrac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)$$

Prop 1. $(|\psi_{xy}\rangle)_{x,y=0,1}$ is an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$

Proof Ex : $\langle \psi_{00} | \psi_{01} \rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|10\rangle + |01\rangle)$

$= \frac{1}{2}(\langle 0|1\rangle\langle 0|0\rangle + \langle 0|0\rangle\langle 0|1\rangle + \langle 1|1\rangle\langle 1|0\rangle + \langle 1|0\rangle\langle 1|1\rangle)$

$= 0$

$\langle \psi_{00} | \psi_{10} \rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle - |11\rangle)$

$= \frac{1}{2}(\langle 0|0\rangle\langle 0|0\rangle - \langle 0|1\rangle\langle 0|1\rangle + \langle 1|0\rangle\langle 1|0\rangle - \langle 1|1\rangle\langle 1|1\rangle)$

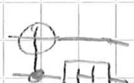
$= \frac{1}{2}(1-1) = 0$ $\boxtimes$

then Bob's operation recovers x, y from $|\psi_{xy}\rangle$

Prop 2. $U' |\psi_{xy}\rangle = |xy\rangle$.

abstractly we are guaranteed to have such U' by Prop 1

Proof. we check $\langle x'y' | U' | \psi_{xy} \rangle = \delta_{xx'}\delta_{yy'}$

$U' =$  $= (I_2 \otimes H)\left(\underset{\bullet}{\overset{\oplus}{|}}\right)$, $(I_2 \otimes H)^\dagger = I_2^\dagger \otimes H^\dagger = I_2 \otimes H$

$\rightsquigarrow$ left hand side is $\overbrace{(\underbrace{(I_2 \otimes H)}_{\text{unitary}} |x'y'\rangle)^\dagger}^{\text{ONB}} \overbrace{(\underbrace{\underset{\bullet}{\overset{\oplus}{|}}}_{\text{unitary}}) |\psi_{xy}\rangle}^{\text{ONB}}$

$\rightsquigarrow$ we want to check $(I_2 \otimes H)|x'y'\rangle = \left(\underset{\bullet}{\overset{\oplus}{|}}\right)|\psi_{xy}\rangle$

$(I_2 \otimes H)|x'y'\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |01\rangle), \frac{1}{\sqrt{2}}(|10\rangle \pm |11\rangle)$

$= \underset{\bullet}{\overset{\oplus}{|}} |\psi_{xy}\rangle$.

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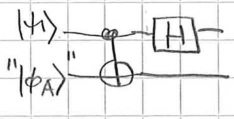
Quantum teleportation (§ 10.1)

basic premise

we can send a qubit through an entangled state. and classical communication

how it works

- Alice wants to send a qubit $|\psi\rangle$ to Bob.
- they prepare a max. ent. st $|\phi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ beforehand
- Alice performs unitary transf. on her state "$|\psi\rangle \otimes |$first factor of $\phi\rangle$", then measure both qubit in $|0\rangle, |1\rangle$
- Alice communicates the result to Bob
- Bob performs a unitary transform on his state $\rightsquigarrow$ it becomes $|\psi\rangle$

Alice's op. 

Bob's op. $U = U_1 U_0$

if Alice's meas. for the first qubit (factor of $|\psi\rangle$) was 0, $U_0 = I_2$ if it was 1, $U_0 = X$

if her meas. for the second qubit (factor of $|\phi_A\rangle$) was 0, $U_1 = I_2$ if it was 1, $U_1 = Z$

why this works : write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

they start with $|\psi\phi\rangle = |\psi\rangle \otimes |\phi\rangle$

$$= \frac{1}{\sqrt{2}}\left(\alpha(|000\rangle + |011\rangle) + \beta(|100\rangle + |111\rangle)\right)$$

Alice's op.

$$|00\rangle, |01\rangle \xrightarrow{CNOT} |00\rangle, |01\rangle \xrightarrow{H \otimes I_2} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

$$|10\rangle, |11\rangle \longrightarrow |11\rangle, |10\rangle \longrightarrow \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$

So the st is $|\tilde{\psi}\rangle = \frac{1}{\sqrt{2}}\left(\alpha(|0**\rangle + |1**\rangle) + \beta(|0**'\rangle - |1**'\rangle)\right)$

$**'$

(cont.) after Alice observes xy, the state is (Bob's)

normalization of $((\langle xy| \otimes I_2)|\tilde{\psi}\rangle)$

$$\left.\begin{array}{lcl} x,y=0,0 & : & \alpha|0\rangle+\beta|1\rangle \\ 0,1 & : & \alpha|1\rangle+\beta|0\rangle \\ 1,0 & : & \alpha|0\rangle-\beta|1\rangle \\ 1,1 & : & \alpha|1\rangle-\beta|0\rangle \end{array}\right\} |\tilde{\psi}_B\rangle$$

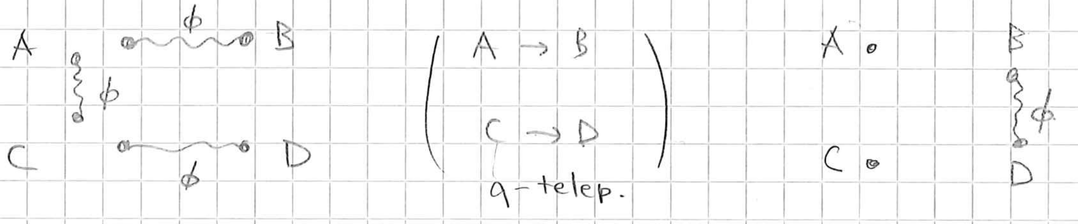
Then Bob uses x, y to correct $|\tilde{\psi}_B\rangle$ to $|\psi\rangle$

More formal formulation: the map

$$|\psi\rangle \mapsto ((\langle 00|\otimes I_2+\langle 01|\otimes Z+\langle 10|\otimes X+\langle 11|\otimes ZX)(H\otimes I_2)(CNOT)(|\psi\rangle\otimes|\phi\rangle)$$

is equal to identity $|\psi\rangle \mapsto |\psi\rangle$.

Entanglement swapping



$$\text{i.e.}\quad \underset{\phi}{\overset{B}{\mathbb{C}^2}} \otimes \overbrace{\mathbb{C}^2\otimes\mathbb{C}^2}^{A} \otimes \overbrace{\mathbb{C}^2\otimes\mathbb{C}^2}^{C}\otimes\overset{D}{\mathbb{C}^2} \longrightarrow \overset{B}{\mathbb{C}^2}\otimes\overset{D}{\mathbb{C}^2}$$

$$\phi \qquad \phi \qquad \phi \qquad \longmapsto \qquad \phi$$

by doing $((\langle 00|\otimes I_2 + \cdots)(H\otimes I_2)\,CNOT$

on the first three factors (with factors rearranged)

and the last three factors.