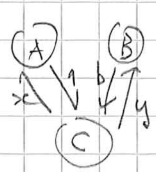
Nonlocal games (section 13.2 & lecture 14)

We use game theory paradigm to understand quantum mechanical phenomena.



basic format

- two cooperative players Alice & Bob
- each receives some input, from Charlie, and produces output
  but no communication after receiving input
- goal : produce particular combination of outputs that satisfy ("win") some conditions.

The Clauser-Horne-Shimony-Holt game

- Alice : input $x = 0, 1$, output $a = 0, 1$
- Bob : ~ $y = 0, 1$, ~ $b = 0, 1$

winning condition : $a + b \equiv xy \mod 2$

i.e. $x = 1 = y \Rightarrow a = 0, b = 1$ or $a = 1, b = 0$

otherwise $\Rightarrow a = 0, b = 0$ or $a = 1, b = 1$.

Classical strategy (example)

$a = 0 = b$ regardless of x, y

(Alice can only use x to calculate a, etc.)

winning prob. 75 % (optimal classical str.)

Quantum strategy

Alice and Bob prepare the maximally entangled state

$\phi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Alice can transform / measure the first factor

Bob ~ second ~

- Alice : if $x = 0$ then do measurement in $|0\rangle, |1\rangle$

  obs $|0\rangle \Rightarrow a = 0$, $|1\rangle \Rightarrow a = 1$

  if $x = 1$ then do meas. in $|+\rangle, |-\rangle$

  obs $|+\rangle \Rightarrow a = 0$, $|-\rangle \Rightarrow a = 1$.

(Cont.)

- Bob : if $y=0$ then do meas. in the basis $|\frac{\pi}{8}\rangle = R_{\theta=\frac{\pi}{8}}|0\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$

$$|\tfrac{5\pi}{8}\rangle = R_{\theta=\frac{5\pi}{8}}|0\rangle = R_{\theta=\frac{\pi}{8}}|1\rangle = \cos\frac{5\pi}{8}|0\rangle + \sin\frac{5\pi}{8}|1\rangle$$

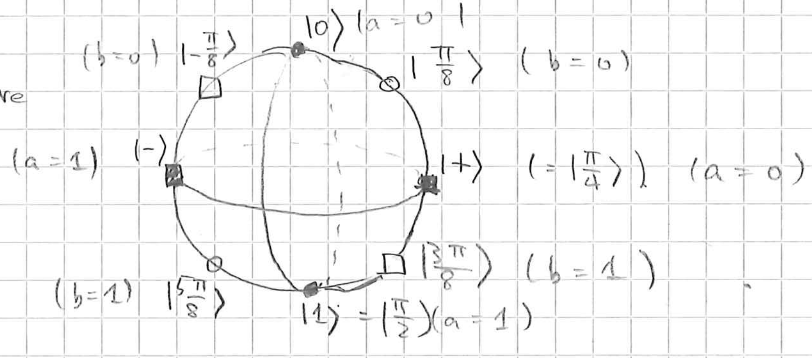
obs $|\frac{\pi}{8}\rangle \Rightarrow b=0$, $|\frac{5\pi}{8}\rangle \Rightarrow b=1$

if $y=1$ then do meas. in the basis $|-\frac{\pi}{8}\rangle = \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle$

$$|\tfrac{3\pi}{8}\rangle = \cos\frac{3\pi}{8}|0\rangle + \sin\frac{3\pi}{8}|1\rangle$$

obs $|-\frac{\pi}{8}\rangle \Rightarrow b=0$, $|\frac{3\pi}{8}\rangle \Rightarrow b=1$

Bloch sphere



Estimate of winning prob.

Suppose $x=0$ and Alice obs. $|0\rangle$ then Bob's state collapses to $|0\rangle$

if $y=0$ $P[\text{Bob}: |\frac{\pi}{8}\rangle] = |\langle\frac{\pi}{8}|0\rangle|^2 = \cos^2\frac{\pi}{8}$ $(b=0)$

$P[\text{Bob}: |\frac{5\pi}{8}\rangle] = |\langle\frac{5\pi}{8}|0\rangle|^2 = \cos^2\frac{5\pi}{8}$ $(=\sin^2\frac{\pi}{8})$

if $y=1$ $P[\text{Bob}: |-\frac{\pi}{8}\rangle] = |\langle-\frac{\pi}{8}|0\rangle|^2 = \cos^2\frac{\pi}{8}$ $(b=0)$

$P[\text{Bob}: |\frac{3\pi}{8}\rangle] = |\langle\frac{3\pi}{8}|0\rangle|^2 = \cos^2\frac{5\pi}{8}$ $(b=1)$

win : $b=0$ ⇝ $\cos^2\frac{\pi}{8}$ prob. for each case

similar computation for all other cases.

⇝ Regardless of Charlie's strategy, Alice & Bob win with prob. $\cos^2\frac{\pi}{8} \sim 85\%$

So (winning prob. from q-scheme) > (optimal winning prob. from classical scheme)

(variant of) <u>Bell's</u> <u>inequality</u>, verified (~ 2016)

Tsirelson's inequality. (§ 14.1.3)

$\exists$ universal const (real Grothendieck const.)
$1.67\cdots < K_G^{\mathbb{R}} < 1.78\cdots$ s.t.

(opt. winning prob. from q-scheme) $\leq K_G^{\mathbb{R}}$ (opt. win. prob. from classical scheme)

more concretely

$LC_{m,n} = \{ (\mathbb{E}[X_i Y_j])_{0 \leq i < m, 0 \leq j < n} : X_i, Y_j$ rand. var $-1 \leq X_i, Y_j \leq 1 \}$

$QC_{m,n} = \{ (\mathrm{Tr}(\rho A_i B_j))_{0 \leq i < m, 0 \leq j < n} : \rho$: density mat. A_i, B_j selfadj, $A_i B_j = B_j A_i$, $\|A_i\|, \|B_j\| \leq 1 \}$

$QC_{m,n} \subset K_G^{\mathbb{R}} \cdot LC_{m,n}$.

For the CHSH game the above q-str. is opt.

$\underset{0.85}{(\text{q-win prob.})} \leq 1.14 \underset{0.75}{(\text{c-win prob.})}$

- Odd cycle game (§ 14.2)

$n = 2k+1$ odd number

- Alice : input $0 \leq x < n$, a = RED, BLUE
- Bob : input $0 \leq y < n$, $b = \sim$.
- Charlie : by prob $\frac{1}{2}$ gives $(x, y) = (i, i)$.
  $\rightsquigarrow$ Alice & Bob should respond with same color.
  by prob $\frac{1}{2}$ gives $(x, y) = (i, j)$, $j = i \pm 1$ (mod n)
  $\rightsquigarrow$ Alice & Bob should respond with different color.

Classically : Alice and Bob can fix alternating colors for inputs $0, 1, \cdots, n-2$.
R, B, R, .. , B
(and do whatever for the input $n-1$)
$\rightsquigarrow$ winning prob. $1 - \frac{1}{2n}$

Quantum strategy : share $\phi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Alice : given input x, do a meas. in the

basis $|\frac{x\pi}{2n}\rangle, \; |\frac{\pi}{2} + \frac{x\pi}{2n}\rangle$ $\quad (R_{\theta=\frac{\pi}{2n}}|0\rangle, \; R_{\theta=\frac{\pi}{2n}}|1\rangle)$

reply RED BLUE (x ev) BLUE, RED (x odd)

Bob : same with input y.

If Alice observes $|\frac{x\pi}{2n}\rangle$, Bob's state collapses

to (normalization of) $(\langle\frac{x\pi}{2n}| \otimes I_2)(|00\rangle + |11\rangle)$

$= \cos\frac{x\pi}{2n}|0\rangle + \sin\frac{x\pi}{2n}|1\rangle = |\frac{x\pi}{2n}\rangle$

when they receive the same input (prob $\frac{1}{2}$)

they are guaranteed to answer the same color

when they receive different inputs (prob $\frac{1}{2}$)

Bob measures $|\frac{x\pi}{2n}\rangle$ in $|\frac{(x\pm1)\pi}{2n}\rangle, \; |\frac{\pi}{2} + \frac{x\pm1}{2n}\rangle$

or $|\frac{\pi}{2} + \frac{x\pi}{2n}\rangle$

$\rightsquigarrow$ losing prob. $\sin^2\frac{\pi}{2n} = O(\frac{1}{n^2})$

winning prob $1 - O(\frac{1}{n^2})$