Hidden variable theories (lect. 13).

What does the CHSH game really say about quantum mechanics ?

⇝ it gives refutation of (local) <u>hidden variable theories</u>.

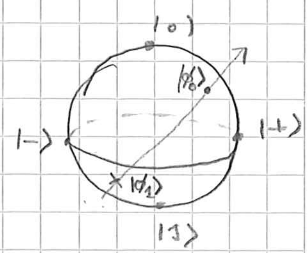
Basic idea behind hidden var. theory :

suppose independent measurements by Alice and Bob are correlated ⇝ there could be some "hidden" parameter that they cannot measure giving correlations

Suppose Alice and Bob measure qubit states

Alice can freely choose basis $|\phi_0\rangle, |\phi_1\rangle$ for measurement

⇝ axis through the Bloch splere.
or point "$|\phi_0\rangle$" in S^2



result can be represented by

$1 \leftrightarrow |\phi_0\rangle$, $-1 \leftrightarrow |\phi_1\rangle$.

Same for Bob

(More concretely, they can measure spin of electron (up / down) in some direction.

If there is a hidden var. z, Alice's choice of meas. and corresp. outcome can be modeled as a func. $A(p, z) = \pm 1$, $p \in S^2$

same for Bob $B(p', z) = \pm 1$, $p' \in S^2$

If we want to talk about expectation

z represents a variable on a probability space (Ω, P).

⇝ for fixed p, $A(p, z)$ is a rand. var on Ω

(cont.) exp. of Alice's meas.: $\mathbb{E}[X_p] = \int X_p(z)\, d\mu_{\mathbb{P}}(z)$

for $X_p(z) = A(p, z)$. $\mu_{\mathbb{P}}$: measure for $\mathbb{P}$

same for Bob. $Y_{p'}(z) = B(p', z)$

In the CHSH game:

Alice and Bob wins if $A(p_x, z)\, B(p'_y, z) = (-1)^{xy}$

$(x, y) = (1, 1)$ ⤳ they need $(a, b) = (0, 1)$ or $(1, 0)$

i.e. $AB = -1$.

otherwise they need $(a, b) = (0, 0)$ or $(1, 1)$

i.e. $AB = 1$.

⤳ winning prob. would be

$$\sum_{x,y=0,1} p_{xy}\ \mathbb{E}\left[\frac{1}{2}\left(\underbrace{X_{p_x} Y_{p'_y} (-1)^{xy}}_{1 \to \text{win},\ -1 \to \text{lose}} + 1\right)\right]$$

when $p_{xy} = \frac{1}{4}$ this gives

$$\mathbb{P}[\text{win}] = \frac{1}{8}\ \mathbb{E}\left[X_0 Y_0 + X_1 Y_0 + X_0 Y_1 - X_1 Y_1\right] + \frac{1}{2}$$

$X_i(z), Y_i(z) = \pm 1 \Rightarrow (X_0 Y_0 + \cdots - X_1 Y_1)(z) = -2, -1, 0, 1, 2$

$\Rightarrow |\mathbb{E}[\cdots]| \leq 2 \Rightarrow \mathbb{P}[\text{win}] \leq \frac{3}{4}$

But the quantum scheme gives winning prob

$\cos^2 \frac{\pi}{8} \sim 0.85 > \frac{3}{4}$ (Bell-CHSH inequality)