Universality for quantum gates, continued (lect. 16)

see also : Nielsen - Chuang §4.5

- factorization of unitary gates

Prop 1. U : $d\times d$ - unitary matrix

then $\exists$ unitary matrices $V_0, \dots, V_{k-1}$ s.t.

- each V_i is concentrated in two rows & columns

$$V_i = \begin{array}{c} \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & a & & b & \cdots & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & c & & d & \cdots & \\ & & \vdots & & \vdots & 1 & \\ & & & & & & \ddots \end{bmatrix} \\ \quad\quad j \qquad\ k \end{array} \begin{array}{l} \\ \\ j \\ \\ \\ k \\ \\ \\ \end{array}$$

- $U = V_0 \cdots V_{k-1}$

Variation : in the above factorization, we can choose

$$V_i = \begin{array}{c} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & b & \cdots \\ & & c & d & \cdots \\ & & \vdots & \vdots & 1 \\ & & & & \ddots \end{bmatrix} \\ \qquad j_i \quad j_i+1 \end{array} \begin{array}{l} \\ \\ j_i \\ j_i+1 \\ \\ \\ \end{array}$$

Proof 1 (for Variation, but inefficient)

strategy : find a <u>neighborhood</u> S of I_d in $U(d)$.

s.t. $U' \in S \Rightarrow \exists\, V_0, \dots, V_{m-1}$ as above
$U' = V_0 \cdots V_{m-1}$.

(then any $U \in U(d)$ is of the form $U = U'_0 \cdots U'_{\ell-1}$
for some $U'_i \in S$.

$\mathfrak{u}_d = \{ X = (x_{ij})_{0\le i,j<d} \in M_d,\ \underbrace{X^\dagger = -X}_{\text{skew-Hermitian}} \}$

$\mathfrak{u}_{2,i} = \{ X \in \mathfrak{u}_d,\ x_{j,k} = 0$ unless $j,k \in \{i, i+1\} \}$

$$\mathfrak{u}_{2,0} = \left\{ \begin{bmatrix} x_{00} & x_{01} & 0 & \cdots \\ x_{10} & x_{11} & 0 & \cdots \\ 0 & 0 & 0 & \\ & & & \ddots \end{bmatrix} \right\},\quad \mathfrak{u}_{2,1} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x_{11} & x_{12} & 0 \\ 0 & x_{21} & x_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

Step 1 $\mathfrak{u}_{2,i}$ $(0 \le i < d-1)$ generate $\mathfrak{u}_d$ as a Lie algebra.

i.e. · linear comb. $\alpha Y + \beta Z$ $\quad \alpha, \beta \in \mathbb{R}$

· bracket $[Y, Z] = YZ - ZY$

on the elements of $\mathfrak{u}_{2,i}$ can produce any elem. in $\mathfrak{u}_d$

ex. $\left[\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Step 2. $\exp(Y) = I_d + Y + \frac{1}{2}Y^2 + \cdots$ $\quad (Y \in \mathfrak{u}_d)$

satisfies · $\exp(Y) \in U(d)$

· $\exp(Y)\exp(Z) \sim \exp(Y+Z)$ ← for small Y, Z

· $\exp(Y)\exp(Z)\exp(-Y)\exp(-Z) \sim \exp([Y, Z])$

⇒ products of matrices of the form $\exp(Y)$ $(Y \in \mathfrak{u}_{2,i})$

can hit arbitrary $\exp(X)$ for small $X \in \mathfrak{u}_d$

$S = \{\exp(X) : X$ such small elem. in $\mathfrak{u}_d\}$

Proof 2 (algorithmic)

Strategy : given $U \in U(d)$, find $U_0, \cdots, U_{k-1}$ s.t.

· each U_i is concentrated in two rows & columns

· $U_{k-1} U_{k-2} \cdots U_0 U = I_d$ $\quad (\Leftrightarrow U = U_0^\dagger U_1^\dagger \cdots U_{k-1}^\dagger)$

Step 1 find U_0 s.t. $U_0 U = \begin{bmatrix} u'_{00} & u'_{01} & \cdots \\ 0 & u'_{11} & \cdots \\ u'_{20} & \cdots & \\ \vdots & & \end{bmatrix}$

write $U = \begin{bmatrix} a & \cdots \\ c & \cdots \\ \vdots & \ddots \end{bmatrix}$ $\quad$ if $c = 0$, we can take $U_0 = I_d$

otherwise $U_0 = \begin{bmatrix} \frac{a^*}{\sqrt{|a|^2+|c|^2}} & \frac{c^*}{\sqrt{|a|^2+|c|^2}} & 0 & \cdots \\ \frac{c}{\sqrt{|a|^2+|c|^2}} & \frac{-a}{\sqrt{|a|^2+|c|^2}} & 0 & \cdots \\ 0 & 0 & 1 & \\ & & & \ddots \end{bmatrix}$

$U_0^\dagger U_0 = I_d$, $\quad U_0 U = \begin{bmatrix} u'_{00} & * \\ 0 & \\ * & \end{bmatrix}$

Step 2 find U_1 s.t. $U_1 U_0 U = \begin{bmatrix} u''_{00} & * \\ 0 & * \\ 0 & * \\ u''_{30} & \end{bmatrix}$

similar manipulation with u'_{00} & u'_{20}

$$U_1 = \begin{bmatrix} \frac{u'^{*}_{00}}{\sqrt{|u'_{00}|^2 + |u'_{20}|^2}} & 0 & \frac{u'^{*}_{20}}{\sqrt{\cdots}} \\ 0 & 1 & 0 \\ \frac{u'_{20}}{\sqrt{\cdots}} & 0 & \frac{-u'_{00}}{\sqrt{\cdots}} \end{bmatrix}$$

Step $d-1$ now we have $U_{d-2} \cdots U_0 U = \begin{bmatrix} \tilde{u}_{00} & * \\ 0 & * \\ \vdots & \vdots \\ 0 & \\ \tilde{u}_{d-1,0} & * \end{bmatrix}$

if $\tilde{u}_{d-1} = 0$ then we know $|\tilde{u}_{00}| = 1$

use $U_{d-1} = \begin{bmatrix} \tilde{u}^{*}_{00} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

if $\tilde{u}_{d-1} \neq 0$ use $U_{d-1} = \begin{bmatrix} \frac{\tilde{u}^{*}_{00}}{\sqrt{|\tilde{u}_{00}|^2 + |\tilde{u}_{d-1,0}|^2}} & 0 \cdots 0 & \frac{\tilde{u}^{*}_{d-1,0}}{\sqrt{\cdots}} \\ 0 & & \\ \vdots & & \\ 0 & & \\ \frac{\tilde{u}_{d-1,0}}{\sqrt{\cdots}} & 0 \cdots 0 & \frac{-\tilde{u}_{00}}{\sqrt{\cdots}} \end{bmatrix}$

in any case we will get $U_{d-1} \cdots U_0 U = \begin{bmatrix} 1 & * \\ 0 & * \\ \vdots & \\ 0 & * \end{bmatrix}$

then $U_{d-1} \cdots U_0 U = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & U' \\ 0 & \end{bmatrix}$ by unitarity

continue with factorization of U'

we will end up with $U_{k-1} \cdots U_0 U = I_d$, $k = \frac{d(d-1)}{2}$ ☒