UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3420 — Quantum Computing

Day of examination: Thursday June 2, 2022

Examination hours: 15.00 – 18.00

This problem set consists of 7 pages.

Appendices: Ingen

Permitted aids: Ingen

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Answer all questions from this part. These are all multiple-choice questions, and you can just state your answer without justification.

Problem 1 (weight 5 points)

Which of the following is the most appropriate description about mathematical foundation of quantum mechanics?

1. We model states of a quantum mechanical system by sequences of nonnegative numbers $p_0, p_1, \dots$ such that $\sum_i p_i = 1$. Transformations of such states are represented by certain matrices with real components.
2. We model states of a quantum mechanical system by unit vectors in Hilbert spaces. Transformations of such states are represented by certain matrices with complex components.
3. None of the above.

Problem 2 (weight 5 points)

Suppose we transform a qubit system by the unitary gate S, then perform another transform by the unitary gate H, with the following matrices:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Which matrix represents the overall composite transform?

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

(Continued on page 2.)

Problem 3 (weight 5 points)

Suppose we have a qubit state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and a measurement in the $|0\rangle, |1\rangle$ basis gave $|0\rangle$ as the outcome. What is the state of the system after the measurement?

1. $|+\rangle$
2. $|0\rangle$
3. None of the above

Problem 4 (weight 5 points)

Suppose we have a qubit state

$$|\psi\rangle = \frac{4}{5}|0\rangle + \frac{3i}{5}|1\rangle .$$

What is the probability of observing $|1\rangle$?

1. $\frac{3}{5}$
2. $\frac{3i}{5}$
3. $\frac{9}{25}$

Problem 5 (weight 5 points)

Measure the state $|\psi\rangle$ from the previous problem in the $|+\rangle$, $|-\rangle$ basis, with $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. What is the probability of observing $|+\rangle$?

1. $\frac{1}{2}$
2. $\frac{4+3i}{5\sqrt{2}}$
3. None of the above

Problem 6 (weight 5 points)

What is the dimension of state space representing a system of three qubits?

1. 3
2. 6
3. None of the above

Problem 7 (weight 5 points)

Consider the following states in two-qubit system:

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle , \quad |\psi_1\rangle = |+-\rangle , \quad |\psi_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Which one is entangled?

(Continued on page 3.)

Problem 8 (weight 5 points)

In this problem we work with a single qubit state space, and use the basis $|0\rangle, |1\rangle$ to represent matrices. Which of the following is the density matrix representing a superposition of states $|+\rangle$ and $|-\rangle$, such that $|+\rangle$ will be observed with probability $\frac{2}{3}$?

$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix}, \qquad C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Problem 9 (weight 5 points)

Suppose that we have a state

$$|\psi\rangle = \frac{1}{2}(|00\rangle + i\,|01\rangle + |10\rangle - |11\rangle)$$

in two-qubit system, and partial measurement of the 0-th qubit of this state, in the $|0\rangle$, $|1\rangle$ basis, gave $|0\rangle$. What is the resulting state in the 1-st qubit?

$$|\psi_0\rangle = \frac{1}{2}(|0\rangle + i\,|1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Problem 10 (weight 5 points)

Which is the most appropriate description about quantum universality?

1. When a collection of gates S has quantum universality, any Boolean operation can be exactly written as a product of gates $U_0 \dots U_{n-1}$ with $U_i \in S$.
2. When a collection of gates S has quantum universality, any unitary gate can be exactly written as a product of gates $U_0 \dots U_{n-1}$ with $U_i \in S$.
3. None of the above.

(Continued on page 4.)

Part II

In this part, justify your answer with appropriate amount of reasoning. Choose two out of the four problems from this part. You may choose more than two to answer, but we only use the best two ones among your answers to decide the score.

Problem 11 (weight 25 points)

Recall the superdense coding scheme, as follows:

1. Alice and Bob share the state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
2. Given $x, y \in \{0, 1\}$, Alice computes the unitary gate $U = U_y U_x$ according to

$$U_x = \begin{cases} I_2 & (x = 0) \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & (x = 1) \end{cases}, \qquad U_y = \begin{cases} I_2 & (y = 0) \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & (y = 1) \end{cases}$$

   and transform her part of the state by U.
3. Alice sends her part of the qubit state to Bob.

11a

What are the possibilities for Bob's state after Alice sends her qubit to him? In what sense is this scheme better than classical communication schemes?

11b

If they share the state $|\phi\rangle = |00\rangle$ at the beginning, can Alice use some choice of U according to x, y to achieve the same kind of communication? If so, give a concrete choice of U. If not, explain why it would not work.

11c

Suppose that they share a state of the form $\cos\epsilon\,|\phi\rangle + \sin\epsilon\,|\psi\rangle$ at the beginning, where $|\psi\rangle$ is a state vector orthogonal to $|\phi\rangle$. Then Alice transforms her qubit using the same rule as usual. What would be the approximate probability of Bob obtaining correct result when $\epsilon \ll 1$?

Problem 12 (weight 25 points)

Recall the CHSH game:

> Charlie (referee) gives $x, y \in \{0, 1\}$ to Alice and Bob (cooperative players) respectively. Alice and Bob return outputs $a, b \in \{0, 1\}$ without communication. They win if $a + b \equiv xy \bmod 2$, lose otherwise.

and the following quantum strategy:

1. Alice and Bob share the state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

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2. If $x = 0$, Alice measures her qubit in the basis $|0\rangle$, $|1\rangle$ to decide a. If $x = 1$, she uses $|+\rangle$, $|-\rangle$.

3. If $y = 0$, Bob measures his qubit in the basis

$$\left|\frac{\pi}{8}\right\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle, \qquad \left|\frac{5\pi}{8}\right\rangle = \cos\frac{5\pi}{8}|0\rangle + \sin\frac{5\pi}{8}|1\rangle$$

   to decide b. If $y = 1$, he uses $\left|-\frac{\pi}{8}\right\rangle$, $\left|\frac{3\pi}{8}\right\rangle$.

12a

What is the winning probability of Alice and Bob under the above strategy? Does it depend on Charlie's strategy?

12b

Suppose that Charlie chooses x and y uniformly random between 0 and 1 respectively. If Alice and Bob use some real random variables $A(x, \omega)$ and $B(y, \omega)$ $(\omega \in \Omega)$ on a probability space $(\Omega, \mathbb{P})$, what would be the optimal probability of Alice and Bob winning? Do we get the same estimate when Charlie has a different strategy?

Hint: relate the probability of their win with

$$\sum_{x',y'=0,1} p_{x'y'}\mathbb{E}\left[\frac{1}{2}(A(x',\omega)B(y',\omega)(-1)^{x'y'} + 1)\right],$$

where $p_{x'y'}$ is the probability of Charlie giving $x = x'$, $y = y'$, and $\mathbb{E}$ denotes the expected value on $(\Omega, \mathbb{P})$.

12c

Suppose that Alice and Bob starts with the state $|\psi\rangle = |00\rangle$, but otherwise follow the same scheme. Do they win with a better probability than a classical scheme as above?

Problem 13 (weight 25 points)

Recall Simon's problem:

> Find $s = (s_i)_{0\leq i<n} \in \{0,1\}^n$ from a function $f\colon \{0,1\}^n \to \{0,1\}^n$ satisfying
>
> $$f(x) = f(y) \Leftrightarrow y = x \oplus s,$$
>
> where $x \oplus s$ is the bitwise XOR operation.

13a

Explain a classical approach to Simon's problem based on the birthday attach method. Give an estimate on typical number of steps required to find s.

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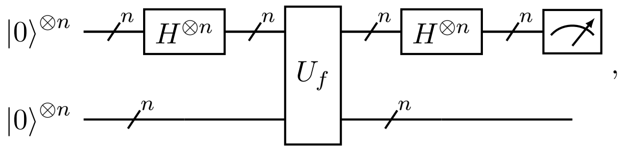
13b

How many $z \in \{0,1\}^n$ satisfying $s \cdot z \equiv 0 \bmod 2$ do we need to know to determine s?

A quantum algorithm finding a z as above can be built using the circuit



where H is the Hadamard gate and U_f is the XOR query $|xy\rangle \mapsto |x(y \oplus f(x))\rangle$. ($-\!\!\!/\!\!-$ denotes several wires bundled together.)

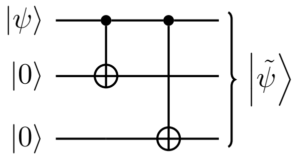
13c

Let $z \in \{0,1\}^n$ be an outcome of the measurement in the above diagram gave $|z\rangle$. Explain why we always have $s \cdot z \equiv 0 \bmod 2$.

Hint: suppose we measured the second half of qubits (left open in the above diagram), and got $|w\rangle$ before applying the second $H^{\otimes n}$. What is the state on the first half of qubits?

Problem 14 (weight 25 points)

Consider the following circuit, that offers protection to the bit flip error by an analogue of 3-bit repetition coding scheme:



14a

Does the construction $|\psi\rangle \mapsto \left|\tilde{\psi}\right\rangle$ violate the no-cloning theorem?

14b

Give an example of a unitary gate $\tilde{H}$ on three qubits satisfying

$$\tilde{H}\left|\tilde{\psi}\right\rangle = \left|\tilde{\phi}\right\rangle, \quad |\phi\rangle = H|\psi\rangle$$

for the Hadamard gate H. You may give it as a matrix, as a quantum circuit, or as a linear map on the three-qubit space.

14c

Suppose that we can use a combination of 15 basic gates (CNOT, H, X, etc.) to encode a qubit as above, transform by $\tilde{H}$, then decode with error correction. Suppose that each of these 15 gates introduce a bit-flip error with

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probability δ. Estimate how small δ should be to have a better accuracy by this scheme than just applying H to the input qubit.

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