MAT3420 2024, Exercises week 12 (for Friday 22 March)

Exercise 1. Exercise 7.1.1 in [1].

Exercise 2 (Quantum teleportation) Let A and B be two laboratories which share the Bell state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and where A wants to send a state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ to B. By the protocol for quantum teleportation, the combined system, before A's measurement, is in the state

$$\left|\tilde{\psi}\right\rangle = \frac{1}{2}\big(|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle + |\psi_4\rangle\big),$$

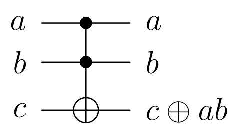
where

$$\begin{aligned}
|\psi_1\rangle &= |00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \\
|\psi_2\rangle &= |01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) \\
|\psi_3\rangle &= |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) \\
|\psi_4\rangle &= |11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle).
\end{aligned}$$

These are the possible states of the system after measurement in the computational basis for A's system, all admitting equal probability 1/4. Compute the density matrix of $\left|\tilde{\psi}\right\rangle$, and show that the reduced density matrix ρ^B is equal to $1/2I$. (This matrix being independent of the state to be teleported is a reason why classical communication is needed to perform the protocol of quantum teleportation.)

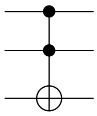
Exercise 3.

(a) Write the table of the reversible "AND" gate below (also known as the Toffoli gate) on all input values to a,b,c in $\{0, 1\}$:



Convince yourself that this gate flips the third bit when both the first and the second bit are 1.

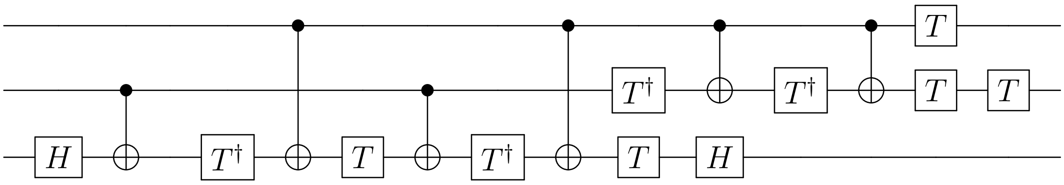
(b) Define now the quantum CCNOT-gate (also known as the Toffoli gate) by the circuit



Recall that T is the 1-qubit gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Show that the CCNOT-gate (a 3-qubit gate) can be implemented as the circuit of single and 2 qubit gates from an universal set of gates as follows (this is Exercise 4.24 in [2]):



Hint: The nontrivial part is to verify the circuit on an input 3-qubit state of the form $|11c\rangle$, with $c \in \{0, 1\}$. Show either by matrix multiplication or by working out the circuit that

$$XTXT^\dagger XTXT^\dagger XH = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

References

[1] P. Kaye, R. Laflamme and M. Mosca, An Introduction to Quantum Computing, Oxford University Press, 2007.

[2] M. Nielsen and Y. Chuang, Quantum Computations and Quantum Information, Cambridge University Press, 7th printing, 2015.