MAT3420 2024, Exercises week 16 (for Friday 19 April)

Exercise 1. Show that the Quantum Fourier Transform is a unitary transformation, as follows. For $m \geq 1$, recall that we defined

$$QFT_m |x\rangle := \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i \frac{xy}{m}} |y\rangle .$$

Convince yourself that with this definition, the matrix F_m corresponding to QFT_m with respect to the standard (computational) basis has entry (x, y), for $0 \leq x, y \leq m-1$ given by

$$[F_m]_{(x,y)} = \frac{1}{\sqrt{m}} e^{2\pi i \frac{yx}{m}} .$$

Then verify that the columns (or, equivalently, the rows) of F_m form an orthonormal set in $\mathbb{C}^m$, by showing that for all $x, x' \in \{0, 1, \ldots, m-1\}$ we have

$$\frac{1}{m} \sum_{y=0}^{m-1} e^{2\pi i \frac{xy}{m}} e^{-2\pi i \frac{x'y}{m}} = \delta_{x,x'}.$$

Exercise 2. Verify the claim on page 119 in [1] that for a given $\omega \in [0, 1)$ and integer $k \geq 2$ the following holds: with probability at least $1 - \frac{1}{2(k-1)}$, the phase estimation algorithm ($QFT_{2^n}^{-1}$) will output of the $2k$ closest integer multiples of $\frac{1}{2^n}$ to ω, by following the steps (after [2]):

(1) Let $x/2^n$ be the best n-bit approximation to ω and let $t := \omega - \frac{x}{2^n}$, where we have assumed that $0 \leq t \leq 2^{-2}$. Consider the amplitude

$$\alpha_z(\omega) := \frac{1}{2^n} \sum_{y=0}^{2^n-1} e^{2\pi i (\omega - \frac{x+z}{2^n}) y} \text{ for each } z = 0, 1, \ldots, 2^n - 1.$$

Split the sum $\sum_{z=0}^{2^n-1} \alpha_z(\omega) |z\rangle$ into two sums, running over $0 \leq z \leq 2^{n-1}$ and $2^{n-1} + 1 \leq z \leq 2^n - 1$; then, in the last sum, change the index of notation to $z' = z - 2^n$, so that we sum over $-2^{n-1} < z' < 0$ (after which z' is relabelled back to z.) Finally, compute each sum of amplitudes $\sum_z |\alpha_z(\omega)|^2$ over the two index sets using the formula for a geometric series $\sum_{y=0}^{2^n-1} q^y$.

(2) For each z, let $\phi_z = 2\pi(t - z/2^n)$, and use that $|1 - e^{i\phi_z}| \geq 2|\phi_z|/\pi$ (because $|\phi_z| \leq \pi$) to conclude that

$$|\alpha_z(\omega)|^2 \leq \frac{1}{4|2^n\phi_z|^2}.$$

(3) Finally estimate $|2^n\phi_z|^2 \geq z^2$ for $-2^{n-1} < z < 0$ and $|2^n\phi_z|^2 \geq (z-1)^2$ for $0 \leq z \leq 2^{n-1}$ to bound the two sums of amplitudes $|\alpha_z(\omega)|^2$ above by $1/2\sum_{z=k}^{2^n-1}\frac{1}{z^2}$, which is the probability of measuring a point $z/2^n$ not among the $2k$ closest integers multiples of $1/2^n$ to ω. The last sum is majorized by $\frac{1}{2(k-1)}$, which proves the claim.

References

[1] P. Kaye, R. Laflamme and M. Mosca, An Introduction to Quantum Computing, Oxford University Press, 2007.

[2] M. Nielsen and Y. Chuang, Quantum Computations and Quantum Information, Cambridge University Press, 7th printing, 2015.