UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2440 — Differential equations and optimal control theory

Day of examination: Thursday 14 June 2018

Examination hours: 9:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Linear systems (weight 15%)

Consider the linear system

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0. \end{cases} \tag{1}$$

For each of the following 2×2 matrices A, classify the equilibrium $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of (1) and draw a phase portrait.

a

$$A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

b

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

c

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Problem 2 Existence and uniqueness (weight 15%)

a

Find a solution of the scalar ODE

$$\begin{cases} \dot{x} = -x^2 \\ x(0) = x_0. \end{cases} \tag{2}$$

(Continued on page 2.)

Determine the maximal interval of existence of (2) for any $x_0 > 0$ — that is, the largest interval $(a, b) \subset \mathbb{R}$ so that a solution $x(t)$ of (2) exists for all $t \in (a, b)$.

b

Consider the scalar ODE

$$\begin{cases} \dot{x}(t) = \sin(x + t) \\ x(0) = x_0. \end{cases} \tag{3}$$

Are there more than one solution of (3)? Justify your answer, for example by applying one of the theorems from the course.

Problem 3 Optimal control (weight 30%)

Consider the optimal control problem

$$\begin{cases} \max \int_0^1 \left(x + \tfrac{1}{2}u\right)e^t \, dt, & \dot{x} = x - u, \\ x(0) = 1, \quad x(1) \text{ is free}, & u(t) \in [0, 1] \text{ for all } t \in [0, 1]. \end{cases} \tag{4}$$

a

Find the adjoint $p(t)$ of the problem.

b

Use the maximum principle to show that

$$u^*(t) = \begin{cases} 0 & \text{if } t < t^* \\ 1 & \text{if } t \geqslant t^* \end{cases}$$

for some $t^* \in (0, 1)$. Find t^*.

c

Find x^*.

d

Show that the candidate (x^*, u^*) is indeed optimal for the problem (4).

Problem 4 Lotka–Volterra (weight 20%)

Consider the Lotka–Volterra model

$$\begin{cases} \dot{n} = n(2 - n - 2m) \\ \dot{m} = m(n - 1) \end{cases} \tag{5}$$

a

Which of the two unknowns represent the predator, and which the prey? Justify your answer.

(Continued on page 3.)

b

The model (5) has an equilibrium at $(n_0^*,\, m_0^*) = \left(1,\, \frac{1}{2}\right)$. Find all the other equilibria.

c

Linearize (5) around the equilibrium $(n_0^*,\, m_0^*)$. Use the linearized system to draw a phase portrait of the nonlinear system (5) around $(n_0^*,\, m_0^*)$. Justify why the linearized system gives a good description of the behavior of (5) close to $(n_0^*,\, m_0^*)$.

Problem 5 (weight 10%)

Consider the ODE

$$\begin{cases} \dot{u} = -v^3 \\ \dot{v} = u^3 \end{cases} \tag{6}$$

Find an expression for the orbits of (6) and draw a phase portrait.

Hint: The system (6) is Hamiltonian.

Problem 6 (weight 10%)

Show that the function $L(u,v) = u^2 + v^2$ is a Lyapunov function for the system

$$\begin{cases} \dot{u} = -v - uv^2 - u^3 \\ \dot{v} = u - v^3. \end{cases} \tag{7}$$

Use this to describe the long-time behavior of the solution (that is, explain what will happen to $x(t) = (u(t), v(t))$ when $t \to \infty$). Justify your answer.

THE END