Problem 1 Linear systems

a

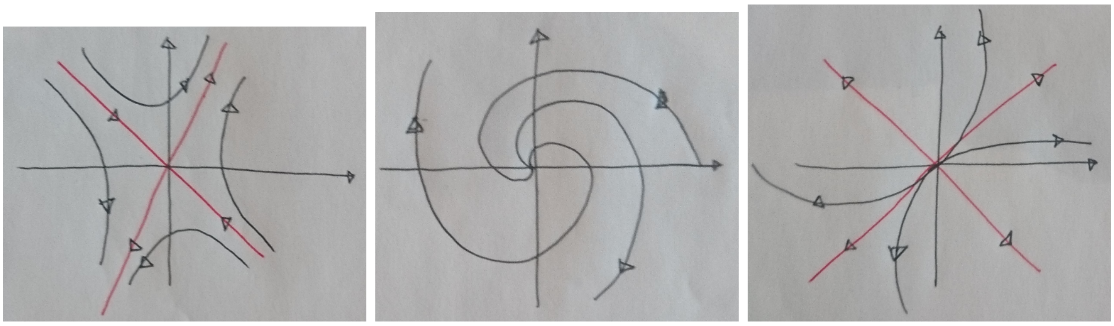
The matrix $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$ is diagonalizable with eigenvector matrix $R = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$. Hence, the equilibrium x^* is a *saddle node.*

b

The matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ is diagonalizable with eigenvector matrix $R = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and eigenvalues $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. Hence, the equilibrium x^* is an *unstable focus* or *source focus.*

c

The matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is diagonalizable with eigenvector matrix $R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Hence, the equilibrium x^* is an *unstable node* or *source node.*



(a) (b) (c)

Figure 1: Phase portraits in Problem 1.

Problem 2 Existence and uniqueness

a

If $x_0 = 0$ then $x(t) \equiv 0$ is a solution. If $x_0 \neq 0$ then $x(t) \neq 0$, at least for small t, so we can divide by x^2 on both sides:

$$\frac{\dot{x}}{x^2} = -1.$$

Integrate with respect to t on both sides:

$$-t = \int_0^t \frac{\dot{x}(s)}{x(s)^2}\, ds = \int_{x_0}^{x(t)} \frac{1}{y^2}\, dy = \frac{1}{x_0} - \frac{1}{x(t)}.$$

Solving for $x(t)$ yields

$$x(t) = \frac{1}{\frac{1}{x_0} + t} = \frac{x_0}{1 + x_0 t}.$$

If $x_0 > 0$ then this solution is well-defined for all $t \in (-\frac{1}{x_0}, \infty)$, but goes to infinity when $t \to -\frac{1}{x_0}$. This interval is therefore the maximal interval of existence.

b

The function $f(x,t) = \sin(x+t)$ is *Lipschitz continuous in the x-variable* (since $|\frac{d}{dx}f(x,t)| = |\cos(x+t)| \leqslant 1$). The uniqueness theorem therefore guarantees that there exists no more than one solution of the ODE.

Problem 3 Optimal control

a

The Hamiltonian for our problem is

$$H(t,x,u,p) = e^t(x + u/2) + p(x-u).$$

Hence, the adjoint p satisfies

$$\dot{p} = -\frac{\partial H}{\partial x} = -e^t - p$$

and $p(1) = 0$. This ODE has solution

$$p(t) = \frac{e^{-t}}{2}\big(e^2 - e^{2t}\big).$$

b

u^* must satisfy

$$\begin{aligned} u^*(t) &= \operatorname{argmax}_{u\in[0,1]} \Big((x + \tfrac{1}{2}u)e^t + (x-u)\frac{e^{-t}}{2}\big(e^2 - e^{2t}\big)\Big) \\ &= \operatorname{argmax}_{u\in[0,1]} \Big(\tfrac{1}{2}ue^t - u\frac{e^{-t}}{2}\big(e^2 - e^{2t}\big)\Big) \\ &= \operatorname{argmax}_{u\in[0,1]} \frac{1}{2}ue^t\Big(2 - e^{2-2t}\Big). \end{aligned}$$

Thus, if $2 - e^{2-2t} \geqslant 0$, i.e. $t \geqslant t^* = 1 - \frac{\ln 2}{2}$, then $u(t) = 1$, and otherwise $u(t) = 0$.

c

x satisfies

$$\dot{x} = \begin{cases} x & \text{if } t < t^* \\ x - 1 & \text{if } t \geqslant t^*, \end{cases} \qquad x(0) = 1,$$

whose solution is

$$x(t) = \begin{cases} e^t & \text{if } t < t^* \\ 1 - e^{t-t^*} + e^t & \text{if } t \geqslant t^*. \end{cases}$$

d

The Hamiltonian H is linear in (x,u) for every t, p, so in particular it is concave with respect to (x,u). It follows from Mangasarian's Theorem that (x^*, u^*) is optimal.

Problem 4 Lotka–Volterra

a

The population m reproduces with the rate $n-1$, so the more n-animals, the faster the increase in m. The population n reproduces with rate $2-n-2m$, so the more m-animals, the slower the increase in n. **We conclude that n is the number of individuals in the prey population, and m in the predator population.**

b

$\dot{m}=0$ requires either $m=0$ or $n=1$, and $\dot{n}=0$ requires either $n=0$ or $n=2-2m$. This yields the two additional equilibria $(n_1^*, m_1^*) = (0,0)$ and $(n_2^*, m_2^*) = (2,0)$.

c

Write $x = \begin{pmatrix} n \\ m \end{pmatrix}$ and $f(x) = \begin{pmatrix} n(2-n-2m) \\ m(n-1) \end{pmatrix}$. The Jacobian of f is

$$Df(x) = \begin{pmatrix} 2-2n-2m & -2n \\ m & n-1 \end{pmatrix}.$$

Hence,

$$A := Df(x_0^*) = \begin{pmatrix} -1 & -2 \\ \frac{1}{2} & 0 \end{pmatrix},$$

and the linearized system is

$$\dot{y} = Ay, \qquad y(0) = x_0 - x_0^*.$$

The eigenvalues of A are $\lambda_\pm = \frac{-1\pm\sqrt{3}i}{2}$, which are complex with $\mathrm{Re}(\lambda_\pm) = -\frac{1}{2} < 0$. It follows that x_0^* is a **stable focus**. To see which direction the flow rotates we can insert, say, $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ into the linearized system, which gives $\dot{y} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ at that specific point. The flow must therefore rotate **counter-clockwise**.

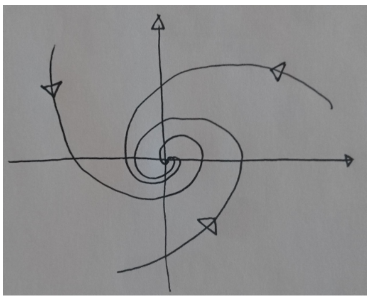


Figure 2: Phase portrait for the linearized system in Problem 4c.

None of the eigenvalues $\lambda_\pm$ has zero real part, so x_0^* is a hyperbolic equilibrium. The Hartman–Grobman theorem then implies that the flow is topologically conjugate to its linearization around x_0^*. We can therefore expect the linearized system to give a good description of the behavior of the ODE close to x_0^*.

Problem 5

The system is Hamiltonian with Hamiltonian function $H(u,v) = \frac{u^4+v^4}{4}$. The solutions of a Hamiltonian system move along the level curves of its Hamiltonian function, so every orbit lie on a curve of the form $\frac{u^4+v^4}{4} = c$ for some constant $c \geqslant 0$. We also note that the system only has one equilibrium, namely $(u^*, v^*) = (0,0)$ (corresponding to $c = 0$), so the solutions never stop anywhere along the curves $\frac{u^4+v^4}{4} = c$ (when $c > 0$). We conclude that the orbits of the system are precisely the curves

$$\frac{u^4+v^4}{4} = c, \qquad c \geqslant 0.$$

Inspecting the vector field anywhere along these curves shows that the solutions move **counter-clockwise** around the origin.

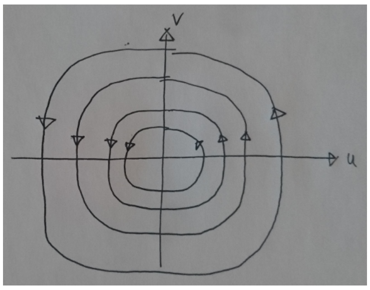


Figure 3: Phase portrait in Problem 5.

Problem 6

Let $(u(t), v(t))$ be any solution of the system and differentiate $L(u,v)$:

$$\frac{d}{dt}L(u,v) = 2u(-v - uv^2 - u^3) + 2v(u - v^3) = -2(u^4 + u^2v^2 + v^4) \leqslant 0,$$

and is only equal to 0 at the equilibrium $(u^*, v^*) = (0,0)$. It follows that L is a Lyapunov function for the equilibrium (u^*, v^*) in the whole of $\mathbb{R}^2$. Hence, (u^*, v^*) attracts all points in $\mathbb{R}^2$, so any solution $(u(t), v(t))$ will converge to $(0,0)$ as $t \to \infty$.