Problem 1 Linear systems

a

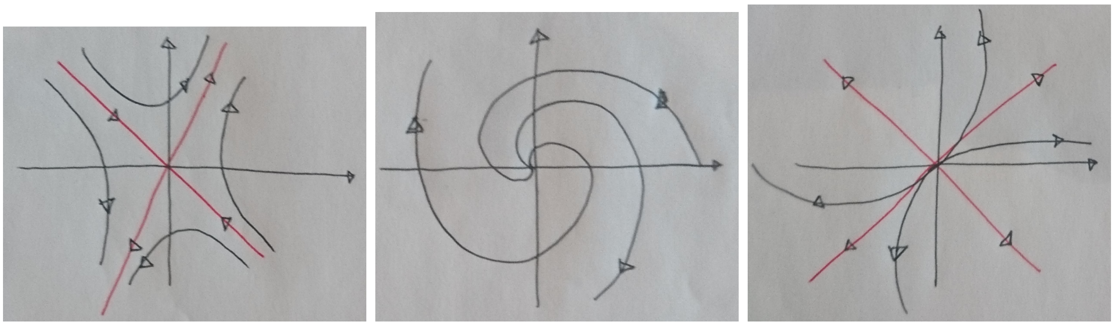
The matrix $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$ is diagonalizable with eigenvector matrix $R = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$. Hence, the equilibrium x^* is a *saddle node.*

b

The matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ is diagonalizable with eigenvector matrix $R = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and eigenvalues $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. Hence, the equilibrium x^* is an *unstable focus* or *source focus.*

c

The matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is diagonalizable with eigenvector matrix $R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Hence, the equilibrium x^* is an *unstable node* or *source node.*



(a) (b) (c)

Figure 1: Phase portraits in Problem 1.

Problem 2 Existence and uniqueness

a

If $x_0 = 0$ then $x(t) \equiv 0$ is a solution. If $x_0 \neq 0$ then $x(t) \neq 0$, at least for small t, so we can divide by x^2 on both sides:

$$\frac{\dot{x}}{x^2} = -1.$$

Integrate with respect to t on both sides:

$$-t = \int_0^t \frac{\dot{x}(s)}{x(s)^2}\, ds = \int_{x_0}^{x(t)} \frac{1}{y^2}\, dy = \frac{1}{x_0} - \frac{1}{x(t)}.$$

Solving for $x(t)$ yields

$$x(t) = \frac{1}{\frac{1}{x_0} + t} = \frac{x_0}{1 + x_0 t}.$$

If $x_0 > 0$ then this solution is well-defined for all $t \in (-\frac{1}{x_0}, \infty)$, but goes to infinity when $t \to -\frac{1}{x_0}$. This interval is therefore the maximal interval of existence.

b

The function $f(x,t) = \sin(x+t)$ is *Lipschitz continuous in the x-variable* (since $|\frac{d}{dx}f(x,t)| = |\cos(x+t)| \leqslant 1$). The uniqueness theorem therefore guarantees that there exists no more than one solution of the ODE.

Problem 3 Numerical methods

a Explicit and implicit Euler

The explicit Euler method is

$$y_{n+1} = y_n - hy_n^2, \qquad n = 0, 1, \ldots.$$

Inserting the prescribed parameters gives

$$y_1 = \frac{9}{8}\left(1 - \frac{1}{8}\cdot\frac{9}{8}\right) = \frac{9}{8}\cdot\frac{55}{64} = \frac{495}{512} \approx \mathbf{0.967}.$$

The implicit Euler method is

$$y_{n+1} = y_n - hy_{n+1}^2.$$

Solving the above for y_{n+1} gives the two solutions

$$y_{n+1} = \frac{-1 \pm \sqrt{1+4hy_n}}{2h}.$$

Assume that $y_n > 0$ (which is the case here). If the solution of the ODE is positive at some time then it stays positive for all times; if we choose the "$-$" solution above then $y_{n+1} < 0$, which would not reflect the behavior of the exact solution. We therefore choose the "+" solution and get

$$y_1 = \frac{-1 + \sqrt{1 + 4\cdot\frac{9}{8}\cdot\frac{1}{8}}}{2\cdot\frac{1}{8}} = -4 + 4\sqrt{\frac{25}{16}} = \mathbf{1}.$$

b Stability of explicit and implicit Euler

The explicit Euler method is linearly stable as long as $|1 + h\lambda_k| \leqslant 1$ for $k = 1, 2$. Since the eigenvalues are real and negative, this is the same as $h \leqslant -\frac{2}{\lambda_k}$ for $k = 1, 2$. Inserting the prescribed eigenvalues gives the requirement

$$h \leqslant \frac{1}{50}$$

to ensure linear stability. The step size $h = \frac{1}{10}$ does not satisfy this requirement, and we can expect large oscillations. The step size $h = \frac{1}{100}$ does satisfy this, and we can expect a reasonable solution.

The implicit Euler method is A-stable (or *unconditionally stable*) and hence gives a reasonable, non-oscillatory solution for any step size $h > 0$.

c Truncation error

Denote the exact solution by $x_n = x(t_n)$ and the truncation error by $\tau_{n+1} = x_{n+1} - z_{n+1}$, where

$$z_{n+1} = x_n + hf\Big(\frac{x_n + z_{n+1}}{2}\Big).$$

From the hint we know that there is a constant $C > 0$ such that

$$\Big|\int_{t_n}^{t_{n+1}} f(x(s))\,ds - hf\Big(\frac{x_n + x_{n+1}}{2}\Big)\Big| \leqslant Ch^3.$$

Using this inequality and the identity

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} f(x(s))\,ds$$

we get

$$|\tau_{n+1}| = \Big|x_n + \int_{t_n}^{t_{n+1}} f(x(s))\,ds - x_n - hf\Big(\frac{x_n + z_{n+1}}{2}\Big)\Big|$$

(adding and subtracting $hf\big(\frac{x_n+x_{n+1}}{2}\big)$)

$$= \Big|\int_{t_n}^{t_{n+1}} f(x(s))\,ds - hf\Big(\frac{x_n + x_{n+1}}{2}\Big) + hf\Big(\frac{x_n + x_{n+1}}{2}\Big) - hf\Big(\frac{x_n + z_{n+1}}{2}\Big)\Big|$$

(triangle inequality)

$$\leqslant \Big|\int_{t_n}^{t_{n+1}} f(x(s))\,ds - hf\Big(\frac{x_n + x_{n+1}}{2}\Big)\Big| + \Big|hf\Big(\frac{x_n + x_{n+1}}{2}\Big) - hf\Big(\frac{x_n + z_{n+1}}{2}\Big)\Big|$$

(by the above estimate and the Lipschitz continuity of f)

$$\leqslant Ch^3 + \frac{1}{2}hK|x_{n+1} - z_{n+1}|$$
$$= Ch^3 + \frac{1}{2}hK|\tau_{n+1}|.$$

Solving for $|\tau_{n+1}|$ gives

$$|\tau_{n+1}| \leqslant \frac{C}{1 - hK/2}h^3.$$

If the time step h is small enough then the coefficient $\frac{C}{1-hK/2}$ is bounded from above. We conclude that the truncation error behaves like h^3.

Problem 4 Lotka–Volterra

a

The population m reproduces with the rate $n-1$, so the more n-animals, the faster the increase in m. The population n reproduces with rate $2-n-2m$, so the more m-animals, the slower the increase in n. **We conclude that n is the number of individuals in the prey population, and m in the predator population.**

b

$\dot{m} = 0$ requires either $m = 0$ or $n = 1$, and $\dot{n} = 0$ requires either $n = 0$ or $n = 2 - 2m$. This yields the two additional equilibria $(n_1^*, m_1^*) = (0, 0)$ and $(n_2^*, m_2^*) = (2, 0)$.

c

Write $x = \begin{pmatrix} n \\ m \end{pmatrix}$ and $f(x) = \begin{pmatrix} n(2 - n - 2m) \\ m(n-1) \end{pmatrix}$. The Jacobian of f is

$$Df(x) = \begin{pmatrix} 2 - 2n - 2m & -2n \\ m & n - 1 \end{pmatrix}.$$

Hence,

$$A := Df(x_0^*) = \begin{pmatrix} -1 & -2 \\ \frac{1}{2} & 0 \end{pmatrix},$$

and the linearized system is

$$\dot{y} = Ay, \qquad y(0) = x_0 - x_0^*.$$

The eigenvalues of A are $\lambda_\pm = \frac{-1 \pm \sqrt{3}i}{2}$, which are complex with $\mathrm{Re}(\lambda_\pm) = -\frac{1}{2} < 0$. It follows that x_0^* is a **stable focus**. To see which direction the flow rotates we can insert, say, $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ into the linearized system, which gives $\dot{y} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ at that specific point. The flow must therefore rotate **counter-clockwise**.

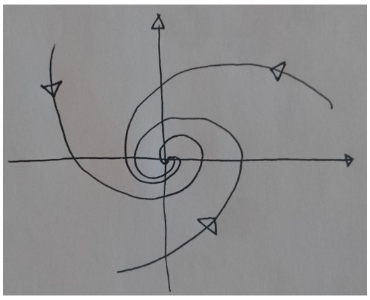


Figure 2: Phase portrait for the linearized system in Problem 4c.

None of the eigenvalues $\lambda_\pm$ has zero real part, so x_0^* is a hyperbolic equilibrium. The Hartman–Grobman theorem then implies that the flow is topologically conjugate to its linearization around x_0^*. We can therefore expect the linearized system to give a good description of the behavior of the ODE close to x_0^*.

Problem 5

The system is Hamiltonian with Hamiltonian function $H(u, v) = \frac{u^4 + v^4}{4}$. The solutions of a Hamiltonian system move along the level curves of its Hamiltonian function, so every orbit lie on a curve of the form $\frac{u^4+v^4}{4} = c$ for some constant $c \geqslant 0$. We also note that the system only has one equilibrium, namely $(u^*, v^*) = (0, 0)$ (corresponding to $c = 0$), so the solutions never stop anywhere along the curves $\frac{u^4+v^4}{4} = c$ (when $c > 0$). We conclude that the orbits of the system are precisely the curves

$$\frac{u^4 + v^4}{4} = c, \qquad c \geqslant 0.$$

Inspecting the vector field anywhere along these curves shows that the solutions move **counter-clockwise** around the origin.

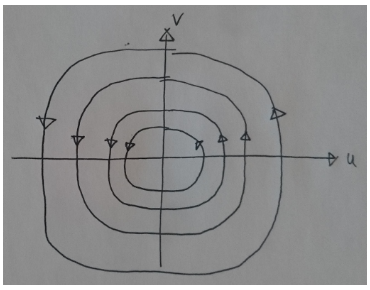


Figure 3: Phase portrait in Problem 5.

Problem 6

Let $(u(t), v(t))$ be any solution of the system and differentiate $L(u, v)$:

$$\frac{d}{dt}L(u, v) = 2u(-v - uv^2 - u^3) + 2v(u - v^3) = -2(u^4 + u^2v^2 + v^4) \leqslant 0,$$

and is only equal to 0 at the equilibrium $(u^*, v^*) = (0, 0)$. It follows that L is a Lyapunov function for the equilibrium (u^*, v^*) in the whole of $\mathbb{R}^2$. Hence, (u^*, v^*) attracts all points in $\mathbb{R}^2$, so any solution $(u(t), v(t))$ will converge to $(0, 0)$ as $t \to \infty$.