UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Friday, June 12th, 2020

Examination hours: 09.00, June 12 – 09.00, June 19

This problem set consists of 5 pages.

Appendices: None

Permitted aids: All aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Solve the differential equation

$$x' = -\frac{1}{1+t}x + 2, \qquad x(0) = 1, \tag{1}$$

by assuming that the solution is of the form

$$x(t) = h(t)i(t),$$

where h solves the homogenous problem and $i(t)$ must be determined.

b

Solve the nonlinear differential equation

$$x' = e^x, \quad x(0) = x_0.$$

What is the maximal time interval $(-\infty, T)$ on which the solution exists? Denote by $\phi_t(x_0)$ the flow. Verify that $\phi_{t+s}(x_0) = \phi_s(\phi_t(x_0))$.

c

Use the "intermediate value theorem" to prove that there exists an equilibrium solution $x^\star$ to the differential equation

$$x' = f(x) = x - \cos(x).$$

Classify the stability of $x^\star$ —unstable (source) or stable (sink). Plot (on a computer) slope fields and phase lines.

(Continued on page 2.)

d

Consider the nonlinear differential equation

$$x' = f_r(x) := r - x^2,$$

which depends on a parameter $r \in \mathbb{R}$. Determine the equilibrium solutions and classify their stability (unstable / stable). Plot (on a computer) slope fields, phase lines, and a bifurcation diagram.

e

Consider the nonlinear differential equation

$$x' = f_r(x) := -x + r\tanh(x),$$

which depends on a parameter $r \in \mathbb{R}$. Discuss the equilibrium solutions and their stability. Make relevant plots on a computer, including slope fields, phase lines, and a bifurcation diagram.

f

Consider the nonlinear differential equation

$$x' = f_r(x) := rx - x^3,$$

depending on a parameter $r \in \mathbb{R}$. Write the equation in gradient form,

$$x' = -\frac{d}{dx}V_r(x),$$

for some function $V_r : \mathbb{R} \to \mathbb{R}$. Plot $V_r(x)$ for different values of r, and determine the local extrema (minima / maxima) of V_r. Prove that V_r decreases along solutions $x(t)$. What is the correspondence between local extrema of V_r and equilibrium solutions of the differential equation (recall the equilibrium solutions and their stability)?

g

Consider the nonlinear differential equation

$$x' = f(x) := -x^3.$$

Why does the linearization method fail to determine the stability of the equilibrium solution 0. Use Liapunov's stability theorem to prove that the origin is stable

Problem 2

a

Show that the approximations defined by Picard iterations converge to the solution $X(t) = \exp(tA)X_0$ of the linear system

$$X' = F(X) = AX, \quad X(0) = X_0, \qquad X_0 \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}.$$

(Continued on page 3.)

b

Use trace–determinant analysis to determine if the linear system $x' = Ax$ has a saddle, (spiral) sink, (spiral) source or center at the origin:

$$\text{(i) } A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \text{(ii) } A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \text{(iii) } A = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix},$$

and

$$\text{(iv) } A = \begin{pmatrix} r & -1 \\ 1 & r \end{pmatrix}, \quad r \in \mathbb{R}.$$

c

Use the matrix exponential to solve the initial–value problem

$$x' = -2x + y, \quad y' = -2y, \qquad x(0) = x_0,\ y(0) = y_0.$$

Sketch the phase portrait.

d

Use the matrix exponential to solve the initial–value problem

$$x' = -2x - y, \quad y' = x - 2y, \qquad x(0) = x_0,\ y(0) = y_0.$$

Sketch the phase portrait.

e

Solve the initial–value problem

$$x' = -2x - y + \cos t, \quad y' = x - 2y + \sin t, \qquad x(0) = 0,\ y(0) = 1.$$

What happens to the solution as $t \to \infty$?

Problem 3

a

Consider the system

$$x' = f(x, y), \qquad y' = g(x, y), \tag{2}$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. Find conditions on f, g such that this system is a gradient system.

Similarly, determine conditions on f, g such that the system is a Hamiltonian system.

b

Consider the nonlinear system

$$x' = \sin(x), \quad y' = -y\cos(x). \tag{3}$$

Explain why this is a Hamiltonian system, and determine the Hamiltonian function $H(x, y)$.

(Continued on page 4.)

c

Find the equilibrium solutions of (3) and use linearization to determine their stability properties. Plot the phase portrait and level sets of the Hamiltonian H (contour plot).

d

We say that the system

$$x' = g(x, y), \qquad y' = -f(x, y) \tag{4}$$

is *orthogonal* to the system (2). Prove that the solution curves of (2) and (4) are orthogonal. Moreover, prove that the orthogonal of a Hamiltonian system is a gradient system.

e

Consider the nonlinear system

$$x' = -y\cos(x), \quad y' = -\sin(x). \tag{5}$$

Explain why this is a gradient system. Find the equilibrium solutions and use linearization to determine their stability properties.

f

Consider the nonlinear system

$$X' = -\nabla V(X),$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Suppose $X^\star$ is a strict local minimum of V. State the Liapunov stability theorem and explain how to use it to conclude that $X^\star$ is a stable equilibrium solution. Provide a definition of "stable" equilibrium solution.

g

Consider problem (f) and the gradient system stated there. Use the method of linearization to prove that the equilibrium solution $X^\star$ is stable.

Problem 4

Consider the nonlinear system

$$x' = x\,(1 - x/a - y)\,, \qquad y' = y\,(x - 1)\,, \tag{6}$$

where $a > 1$ is a constant.

(Continued on page 5.)

a

Determine the nontrivial equilibrium solution $(x^\star, y^\star) \neq (0,0)$. Use the Liapunov stability theorem to prove that $(x^\star, y^\star)$ is asymptotically stable. Provide a definition of "asymptotically stable" equilibrium solution.

Hint: Use

$$V(x,y) = x - \ln x + y + C \ln y, \quad (x,y) \in O := \{x > 0, y > 0\},$$

with an appropriately chosen C, to construct a strict Liapunov function.

b

Consider the nonlinear system (6), and the nontrivial equilibrium solution $(x^\star, y^\star)$. Let $(x(t), y(t))$ be a T-periodic solution of (6). Prove that

$$\langle x \rangle_T := \frac{1}{T}\int_0^T x(t)\,dt = x^\star, \qquad \langle y \rangle_T := \frac{1}{T}\int_0^T y(t)\,dt = y^\star.$$

THE END