UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3440 — Dynamical systems
Day of examination:	Friday, June 12th, 2020
Examination hours:	09.00, June 12 − 09.00, June 19

This problem set consists of 17 pages.

Appendices:	None
Permitted aids:	All aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Solve the differential equation

$$x' = -\frac{1}{1+t}x + 2, \qquad x(0) = 1, \tag{1}$$

by assuming that the solution is of the form

$$x(t) = h(t)i(t),$$

where h solves the homogenous problem and $i(t)$ must be determined.

Answer: Set $a(t) = -1/(1+t)$ and note that

$$\int_0^t a(s)\,ds = -\ln(1+t) = \ln\left(\frac{1}{1+t}\right).$$

The solution of

$$h' = a(t)h, \quad h(0) = 1,$$

is

$$h(t) = \exp\left(\int_0^t a(s)\,ds\right) = \frac{1}{1+t}.$$

Let us guess a solution of (1) of the form

$$x(t) = h(t)i(t),$$

where $i(t)$ must be determined such that $i(0) = 1$. Note that

$$x' = h'i + hi' = a(t)hi + hi'$$

(Continued on page 2.)

and this should equate to

$$a(t)x + 2 = a(t)hi + 2.$$

In other words, we must have $hi' = 2$ or

$$i(t) = \int_0^t \frac{2}{h(t)}\,dt = (1+t)^2.$$

Hence, the solution of (1) is

$$x(t) = h(t)i(t) = 1 + t.$$

b

Solve the nonlinear differential equation

$$x' = e^x, \quad x(0) = x_0.$$

What is the maximal time interval $(-\infty, T)$ on which the solution exists? Denote by $\phi_t(x_0)$ the flow. Verify that $\phi_{t+s}(x_0) = \phi_s\left(\phi_t(x_0)\right)$.

Answer: Separating variables, we write

$$\int e^{-x}\,dx = \int dt$$

or $-e^{-x} = t + C$. Since $x(0) = x_0$, we find $C = -e^{-x_0}$. Solving for x in the equation $e^{-x} = e^{-x_0} - t$ gives

$$x(t) = -\ln\left(e^{-x_0} - t\right).$$

The solution exists as long as $e^{-x_0} - t > 0$, so that $T = e^{-x_0}$.

We have

$$\phi_t(x_0) = -\ln\left(e^{-x_0} - t\right).$$

Note that

$$\phi_{t+s}(x_0) = -\ln\left(e^{-x_0} - t - s\right)$$

and

$$\begin{aligned}\phi_s\left(\phi_t(x_0)\right) = \phi_s\left(-\ln\left(e^{-x_0} - t\right)\right) &= -\ln\left(e^{-\left(-\ln\left(e^{-x_0}-t\right)\right)} - s\right) \\ &= -\ln\left(e^{\ln\left(e^{-x_0}-t\right)} - s\right) = -\ln\left(e^{-x_0} - t - s\right),\end{aligned}$$

and therefore $\phi_{t+s}(x_0) = \phi_s\left(\phi_t(x_0)\right)$.

c

Use the "intermediate value theorem" to prove that there exists an equilibrium solution $x^\star$ to the differential equation

$$x' = f(x) = x - \cos(x).$$

Classify the stability of $x^\star$ —unstable (source) or stable (sink). Plot (on a computer) slope fields and phase lines.

(Continued on page 3.)

Answer: The intermediate value theorem states that if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval $[a, b]$. Clearly $f(x) = x - \cos(x)$ is continuous. Let us take $a = 0$, $f(0) = -1 < 0$ and $b = \pi/2$, $f(\pi/2) = \pi/2 > 0$. Then, by the intermediate value theorem, there exists $x^\star$ such that $f(x^\star) = 0$. Next, we compute $f'(x) = 1 + \sin(x)$ and so $f'(x^\star) > 0$ since $x^\star \in (0, \pi/2)$. Hence, $x^\star$ is unstable. See Figure 1.

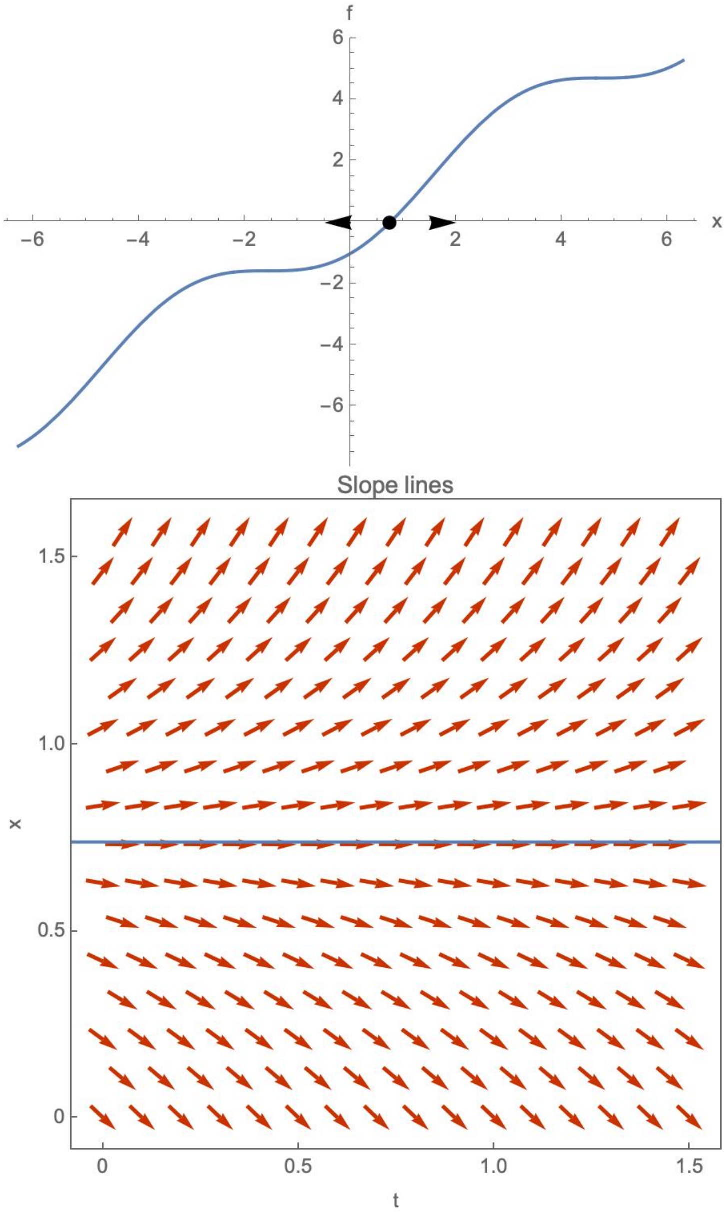

Figure 1: Problem 1–(c). $f(x) = x - \cos(x)$ and phase line (top) and slope field (bottom)

(Continued on page 4.)

d

Consider the nonlinear differential equation

$$x' = f_r(x) := r - x^2,$$

which depends on a parameter $r \in \mathbb{R}$. Determine the equilibrium solutions and classify their stability (unstable / stable). Plot (on a computer) slope fields, phase lines, and a bifurcation diagram.

<u>Answer:</u> The equilibrium solutions are $x^\star = \pm\sqrt{r}$ if $r \geq 0$ and none if $r < 0$. Since $f'(\pm\sqrt{r}) = \mp 2\sqrt{r}$ if $r \geq 0$. Hence, $\sqrt{r}$ is stable, while $-\sqrt{r}$ is unstable. At the bifurcation point $r = 0$, we have $f'(0) = 0$. See Figure 2

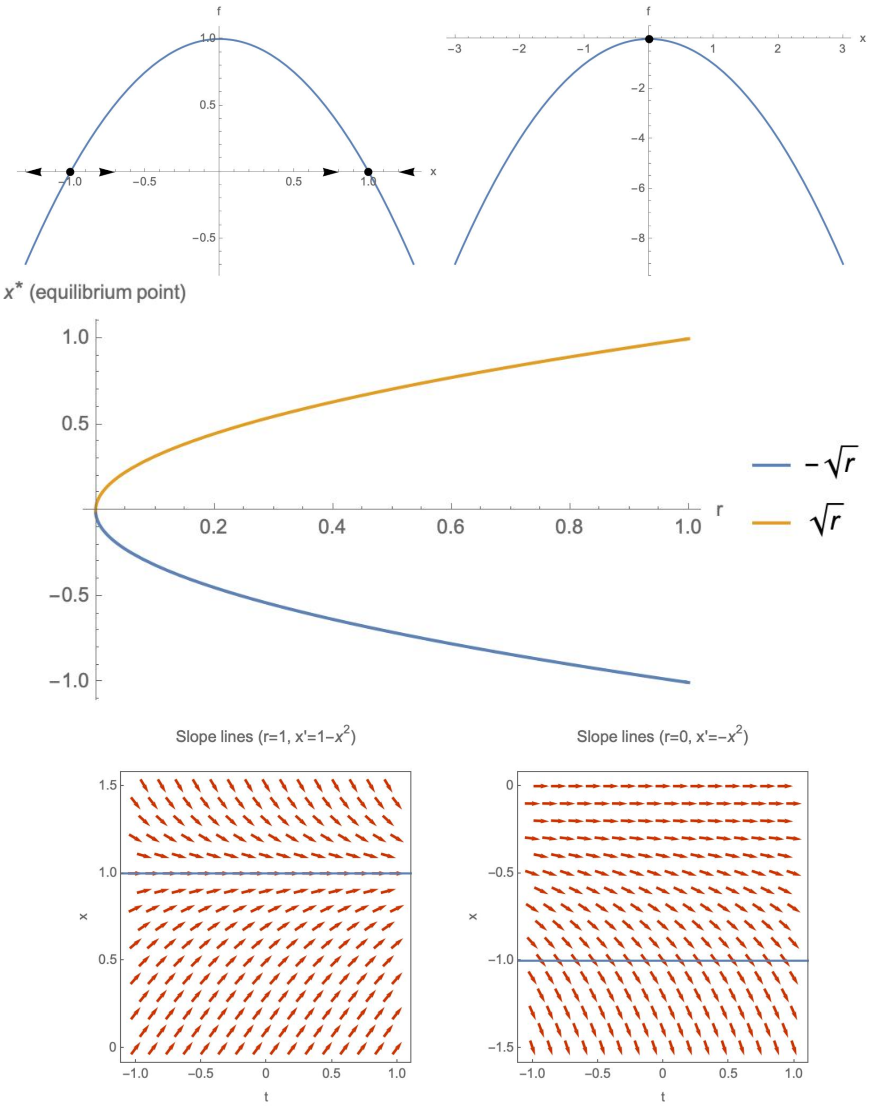

Figure 2: Problem 1–(d). $f(x) = 1 - x^2$ (upper-left, $r = 1$), $f(x) = -x^2$ (upper-right, $r = 0$), bifurcation plot (middle), and slope fields (bottom).

(Continued on page 5.)

e

Consider the nonlinear differential equation

$$x' = f_r(x) := -x + r\tanh(x),$$

which depends on a parameter $r \in \mathbb{R}$. Discuss the equilibrium solutions and their stability. Make relevant plots on a computer, including slope fields, phase lines, and a bifurcation diagram.

<u>Answer:</u> The equilibrium solutions are given by the intersection of the graphs of $y = x$ and $y = r\tanh(x)$. By plotting these graphs for different values of r, we see that there is one equilibrium solution at the origin if $r < 1$. As $f_r'(0) = r - 1$, the origin is stable if $r < 1$. A pitchfork bifurcation occurs at $r = 1$, $x^\star = 0$. For $r > 1$, there are are two additional equilibrium solutions (both stable), while the origin becomes unstable. See Figures 3 and 4.

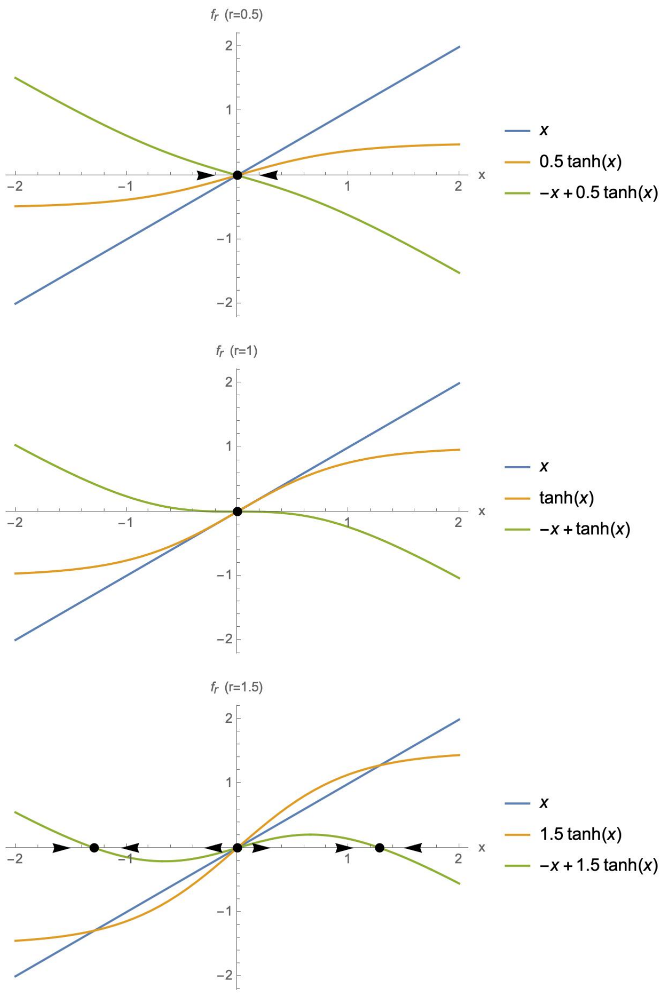

Figure 3: Problem 1-(e).

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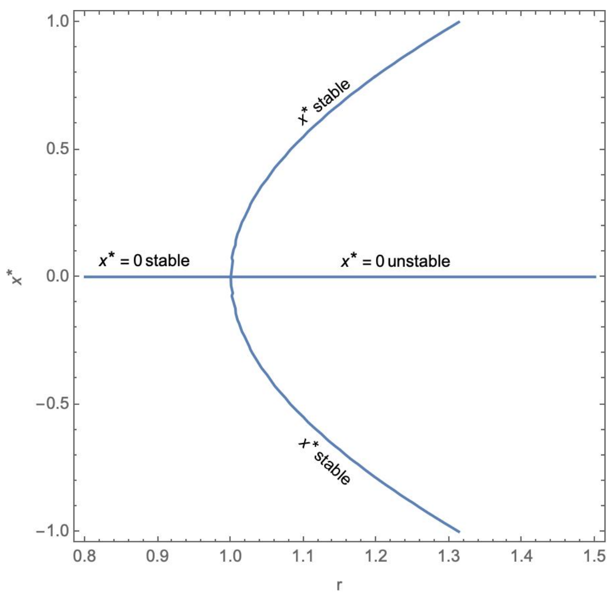

Figure 4: Problem 1-(e). Bifurcation plot.

f

Consider the nonlinear differential equation

$$x' = f_r(x) := rx - x^3,$$

depending on a parameter $r \in \mathbb{R}$. Write the equation in gradient form,

$$x' = -\frac{d}{dx}V_r(x),$$

for some function $V_r : \mathbb{R} \to \mathbb{R}$. Plot $V_r(x)$ for different values of r, and determine the local extrema (minima / maxima) of V_r. Prove that V_r decreases along solutions $x(t)$. What is the correspondence between local extrema of V_r and equilibrium solutions of the differential equation (recall the equilibrium solutions and their stability)?

Answer: Setting

$$V_r(x) = \frac{r}{2}x^2 - \frac{1}{4}x^4,$$

we obtain $x' = -\frac{d}{dx}V_r(x)$. See Figure 5.

If $r \leq 0$, then V_r has a local minimum at $x = 0$. If $r > 0$, then V_r has local minima at $x = \pm\sqrt{r}$ and a local maximum at $x = 0$. Let $x(t)$ be a solution of the differential equation. Then

$$\frac{d}{dt}V_r(x(t)) = V_r'(x(t))x'(t) = -\left(V_r'(x(t))\right)^2 \leq 0.$$

The equilibrium solutions of the equation are $x^\star = 0$ for any r (stable if $r < 0$, unstable if $r > 0$) and $x^\star = \pm\sqrt{r}$ for $r > 0$ (stable), see for example the mandatory assignment. Consequently, a local minimum of V_r corresponds to a stable equilibrium solution and a local maximum corresponds to an unstable equilibrium solution.

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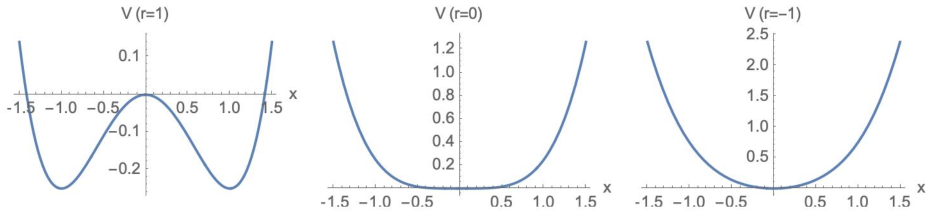

Figure 5: Problem 1-(f).

g

Consider the nonlinear differential equation

$$x' = f(x) := -x^3.$$

Why does the linearization method fail to determine the stability of the equilibrium solution 0. Use Liapunov's stability theorem to prove that the origin is stable

Answer: Since $f'(0) = 0$, the linearization method does not apply. Set $L(x) = \frac{1}{4}x^4$ (motivated by the previous problem). Clearly, $L(0) = 0$ and $L(x) > 0$ for all $x \neq 0$. Moreover,

$$L'(x)f(x) = -x^6 \leq 0.$$

Hence, L is a Liapunov function. The stability of the origin then follows from the Liapunov stability theorem.

Problem 2

a

Show that the approximations defined by Picard iterations converge to the solution $X(t) = \exp(tA)X_0$ of the linear system

$$X' = F(X) = AX, \quad X(0) = X_0, \qquad X_0 \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}.$$

Answer: Consider the Picard iterations $\{u_k(t)\}_{k=0}^{\infty}$ defined by $u_0(t) = x_0$ and

$$u_{k+1}(t) = x_0 + \int_0^t F(u_k(s))\,ds, \quad k = 0, 1, 2, \ldots.$$

For $k \geq 0$, it is easy to check that

$$\begin{aligned} u_k(t) &= X_0 + tAX_0 + \frac{(tA)^2}{2!}X_0 + \ldots + \frac{(tA)^k}{k!}X_0 \\ &= \left(I + tA + \frac{(tA)^2}{2!} + \ldots + \frac{(tA)^k}{k!}\right)X_0, \end{aligned}$$

and therefore

$$\lim_{k\to\infty} u_k(t) = \left(\sum_{I=0}^{\infty} \frac{(tA)^k}{k!}\right) X_0 = \exp(tA)X_0.$$

(Continued on page 8.)

b

Use trace–determinant analysis to determine if the linear system $x' = Ax$ has a saddle, (spiral) sink, (spiral) source or center at the origin:

$$\text{(i) } A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \text{(ii) } A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \text{(iii) } A = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix},$$

and

$$\text{(iv) } A = \begin{pmatrix} r & -1 \\ 1 & r \end{pmatrix}, \quad r \in \mathbb{R}.$$

<u>Answer:</u> For a general matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define

$$D = \det(A) = ad - cd, \quad T = \operatorname{trace}(A) = a + d.$$

(i) We compute

$$D = \det(A) = -2 < 0,$$

which implies a saddle at the origin.

(ii) We compute

$$D = \det(A) = 8 > 0, \quad T = \operatorname{trace}(A) = 6 > 0, \quad T^2 - 4D = 4 > 0,$$

which implies a source at the origin.

(iii) We compute

$$D = \det(A) = 2 > 0, \quad T = \operatorname{trace}(A) = -3 < 0, \quad T^2 - 4D = 1 > 0$$

which implies a sink at the origin.

(iv) We compute

$$D = \det(A) = 1 + r^2 > 0, \quad T = \operatorname{trace}(A) = 2r,$$

and

$$T^2 - 4D = 4r^2 - 4(r^2 + 1) = -4 < 0.$$

If $r > 0$, then $T > 0$ and we have a spiral source at the origin. If $r < 0$, then $T < 0$ and we have a spiral sink at the origin. Finally, if $r = 0$, then $T = 0$ and we have a center at the origin.

c

Use the matrix exponential to solve the initial–value problem

$$x' = -2x + y, \quad y' = -2y, \qquad x(0) = x_0,\ y(0) = y_0.$$

Sketch the phase portrait.

<u>Answer:</u> We write the system in the form

$$X' = AX, \quad X(0) = X_0, \quad A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

(Continued on page 9.)

The solution is

$$X(t) = \exp(tA)X_0.$$

Let us compute $\exp(tA)$. Note that

$$tA = D + B, \quad D = -2tI, \quad B = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Clearly, $DB = BA$ and therefore

$$\exp(tA) = \exp(D)\exp(B),$$

where $\exp(D) = e^{-2t}I$ and, since $B^k = 0$ for $k \geq 2$,

$$\exp(B) = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Consequently,

$$\exp(tA) = e^{-2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and hence

$$x(t) = e^{-2t}(x_0 + y_0 t), \quad y(t) = e^{-2t}y_0.$$

See Figure 6 for the phase portrait.

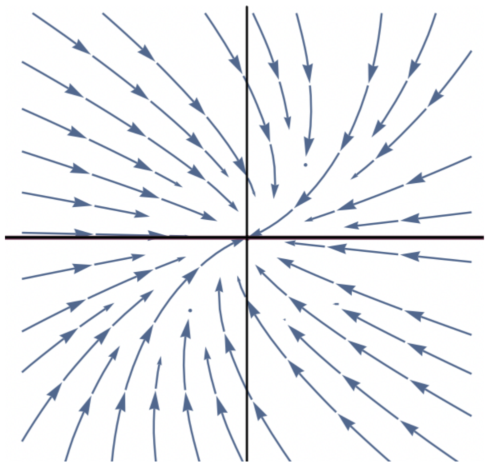

Figure 6: Problem 2-(c). Phase portrait.

d

Use the matrix exponential to solve the initial–value problem

$$x' = -2x - y, \quad y' = x - 2y, \qquad x(0) = x_0,\ y(0) = y_0.$$

Sketch the phase portrait.

<u>Answer:</u> We write the system in the form

$$X' = AX, \quad X(0) = X_0, \quad A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}.$$

(Continued on page 10.)

The solution is

$$X(t) = \exp(tA)X_0.$$

From the mandatory assignment (for example), we know that

$$B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \implies \exp(B) = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.$$

Using this with $a = -2t$ and $b = t$, we find

$$\exp(tA) = e^{-2t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

and thus

$$x(t) = e^{-2t}(x_0 \cos t - y_0 \sin t), \quad y(t) = e^{-2t}\left(x_0 \sin t + y_0 \cos t\right).$$

See Figure 7 for the phase portrait (spiral sink).

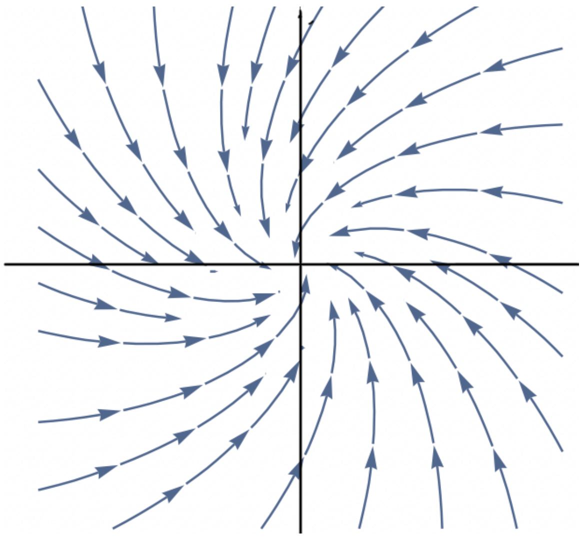

Figure 7: Problem 2-(d). Phase portrait.

e

Solve the initial–value problem

$$x' = -2x - y + \cos t, \quad y' = x - 2y + \sin t, \qquad x(0) = 0,\ y(0) = 1.$$

What happens to the solution as $t \to \infty$?

<u>Answer:</u> We write the system in the form

$$X' = AX + G(t), \quad X(0) = X_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}, \quad G(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

(Continued on page 11.)

By the "variation of parameters" formula, the solution takes the form

$$X(t) = \exp(tA)X_0 + \exp(tA)\int_0^t \exp(-sA)G(s)\,ds.$$

By Problem 2–(d), we have $\exp(tA) = e^{-2t}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$ and $\exp(tA)X_0 = e^{-2t}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$. Moreover,

$$\exp(-sA)G(s) = e^{2s}\begin{pmatrix}\cos s & \sin s\\ -\sin s & \cos s\end{pmatrix}\begin{pmatrix}\cos s\\ \sin s\end{pmatrix} = e^{2s}\begin{pmatrix}1\\ 0\end{pmatrix},$$

$$\int_0^t \exp(-sA)G(s)\,ds = \frac{1}{2}\left(e^{2t}-1\right)\begin{pmatrix}1\\ 0\end{pmatrix},$$

and

$$\begin{aligned}
&\exp(tA)\int_0^t \exp(-sA)G(s)\,ds\\
&\quad = e^{-2t}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\frac{1}{2}\left(e^{2t}-1\right)\begin{pmatrix}1\\ 0\end{pmatrix}\\
&\quad = \frac{1}{2}\left(1-e^{-2t}\right)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}
\end{aligned}$$

Summarizing,

$$\begin{aligned}
X(t) &= e^{-2t}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix} + \frac{1}{2}\left(1-e^{-2t}\right)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + \frac{e^{-2t}}{2}\begin{pmatrix}-\cos t - 2\sin t\\ 2\cos t - \sin t\end{pmatrix}.
\end{aligned}$$

Finally, we observe that

$$\lim_{t\to\infty}\left|X(t) - \frac{1}{2}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\right| = 0.$$

Problem 3

a

Consider the system

$$x' = f(x,y), \qquad y' = g(x,y), \tag{2}$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. Find conditions on f, g such that this system is a gradient system.

Similarly, determine conditions on f, g such that the system is a Hamiltonian system.

Answer: The system (2) is a gradient system,

$$X' = -\nabla V(X), \quad \text{for some } V : \mathbb{R}^2 \to \mathbb{R},$$

(Continued on page 12.)

if and only if

$$\frac{\partial V}{\partial x} = -f, \qquad \frac{\partial V}{\partial y} = -g.$$

Differentiating the first equation with respect to y and the second one with respect to x, we obtain the requirement (since $V_{xy} = V_{yx}$)

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

In other words, the (two-dimensional) curl of the vector field $F = (f, g)$ is zero: $\operatorname{curl} F = g_x - f_y = 0$.

The system (2) is a Hamiltonian system if and only if

$$x' = \frac{\partial H}{\partial y}, \quad y' = -\frac{\partial H}{\partial x}, \quad \text{for some } H : \mathbb{R}^2 \to \mathbb{R}.$$

This translates into the requirement

$$\frac{\partial H}{\partial y} = f, \qquad \frac{\partial H}{\partial x} = -g,$$

and thus (since $H_{yx} = H_{xy}$)

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}.$$

In other words, the vector field $F = (f, g)$ is divergence-free: $\nabla \cdot F = 0$.

b

Consider the nonlinear system

$$x' = \sin(x), \quad y' = -y\cos(x). \tag{3}$$

Explain why this is a Hamiltonian system, and determine the Hamiltonian function $H(x, y)$.

<u>Answer:</u> According to the previous problem with $f(x, y) = \sin(x)$ and $g(x, y) = -y\cos(x)$, we compute the divergence of $F := (f, g)$:

$$\nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \cos(x) - \cos(x) = 0,$$

and so (3) is a Hamiltonian system. The Hamiltonian H satisfies

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} = \cos(x).$$

In view of this (for example), we specify

$$H(x, y) = \sin(x)y.$$

(Continued on page 13.)

c

Find the equilibrium solutions of (3) and use linearization to determine their stability properties. Plot the phase portrait and level sets of the Hamiltonian H (contour plot).

Answer: The equilibrium solutions are given by

$$\sin(x) = 0, \quad y\cos(x) = 0 \quad \iff X_n^\star = (x_n^\star, y_n^\star) = (n\pi, 0),$$

for $n = 0, \pm 1, \pm 2, \ldots$. Set $F = \big(\sin(x), -y\cos(x)\big)$, and compute

$$DF = \begin{pmatrix} \cos(x) & 0 \\ y\sin(x) & -\cos(x) \end{pmatrix}.$$

The linearized system is

$$X' = AX, \quad A = DF(X_n^\star) = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = \pm 1$, one negative and one positive. Hence the equilibrium solutions $X^\star$ are all saddles. See Figure 8.

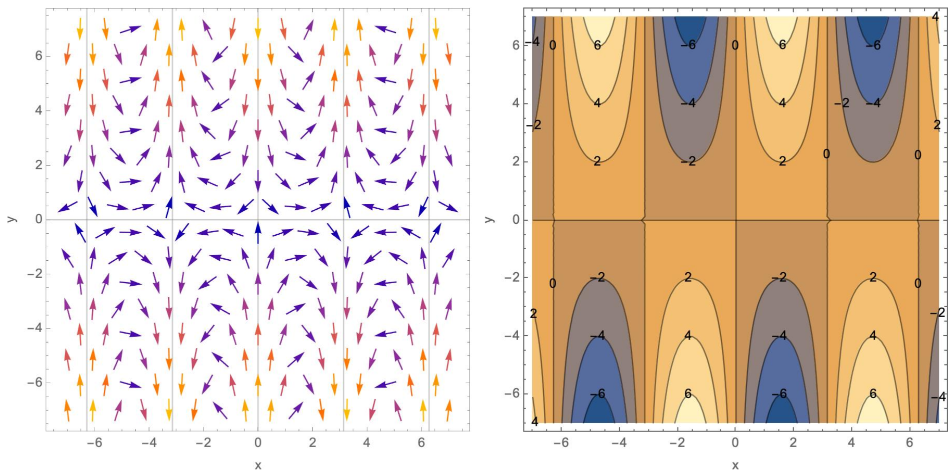

Figure 8: Problem 8–(c). Left: phase portrait of the system (3). Right: contour plot of the Hamiltonian $H(x, y) = \sin(x)y$.

d

We say that the system

$$x' = g(x, y), \qquad y' = -f(x, y) \tag{4}$$

is *orthogonal* to the system (2). Prove that the solution curves of (2) and (4) are orthogonal. Moreover, prove that the orthogonal of a Hamiltonian system is a gradient system.

Answer: Denote by $X(t)$ the solution of (2) and by $X_O(t)$ the solution of (4). Then

$$X'(t) \cdot X_O'(t) = (f, g) \cdot (g, -f) = fg - gf = 0.$$

(Continued on page 14.)

Next, consider a Hamiltonian system

$$x' = \frac{\partial H}{\partial y}, \quad y' = -\frac{\partial H}{\partial x}.$$

The orthogonal of this system is

$$x' = -\frac{\partial H}{\partial x}. \quad x' = -\frac{\partial H}{\partial y}$$

and thus, with $X = (x, y)$ and $V(X) = H(X)$,

$$X' = -\nabla V(X),$$

which is a gradient system.

e

Consider the nonlinear system

$$x' = -y\cos(x), \quad y' = -\sin(x). \tag{5}$$

Explain why this is a gradient system. Find the equilibrium solutions and use linearization to determine their stability properties.

Answer: We apply the previous problem (d), noting that (5) is the orthogonal of the Hamiltonian system (3). Alternatively, we explicitly write

$$X' = -\nabla V(X), \quad X = (x, y), \quad V(X) = \sin(x)y.$$

The equilibrium solutions are given by

$$X_n^\star = (x_n^\star, y_n^\star) = (n\pi, 0),$$

for $n = 0, \pm 1, \pm 2, \dots$. Set $F = \big(-y\cos(x), -\sin(x)\big)$, and compute

$$DF = \begin{pmatrix} y\sin(x) & -\cos(x) \\ -\cos(x) & 0 \end{pmatrix}.$$

The linearized system is

$$X' = AX, \quad A = DF(X_n^\star) = (-1)^n \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = \pm 1$, so the equilibrium solutions are saddles.

f

Consider the nonlinear system

$$X' = -\nabla V(X),$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Suppose $X^\star$ is a strict local minimum of V. State the Liapunov stability theorem and explain how to use it to conclude that $X^\star$ is a stable equilibrium solution. Provide a definition of "stable" equilibrium solution.

(Continued on page 15.)

Answer: By assumption, $X^\star$ is local minimum. This implies $\nabla V(X^\star) = 0$ and thus $X^\star$ is an equilibrium solution. The equilibrium solution $X^\star$ is stable if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|X_0 - X^\star| < \delta \Longrightarrow |X(t) - X^\star| < \varepsilon \quad \forall t \geq 0.$$

The Liapunov stability theorem can be found in the book on page 193. To conclude the stability of $X^\star$ it is enough to exhibit a Liapunov function L, i.e., $L(X^\star) = 0$, $L(X) > 0$ if $X \neq X^\star$, and

$$-\nabla L(X) \cdot \nabla V(X) \leq 0, \quad \forall X \neq X^\star.$$

Since $X^\star$ is a strict local minimum of V, it is easy to check that

$$L(X) = V(X) - V(X^\star)$$

is a Liapunov function (locally around $X^\star$).

g

Consider problem (f) and the gradient system stated there. Use the method of linearization to prove that the equilibrium solution $X^\star$ is stable.

Answer: Set $F(X) = -\nabla V(X)$. Then $-DF(X)$ equals the Hessian of V, i.e., $DF(X) = -D^2V(X)$. The linearized system is

$$X' = AX, \quad A = DF(X_n^\star) = -D^2V(X_n^\star).$$

Note that A is a symmetric matrix, which implies that the eigenvalues of A are real. Recall that $X^\star$ is a local minimum of V. Hence, the Hessian of V at $X^\star$ is positive definite and so A is a negative definite matrix, i.e.,

$$A < 0 \quad \text{(which means } X^\top AX < 0 \; \forall X \neq 0).$$

We recall that the matrix A is negative definite if and only if all of its eigenvalues are negative. Hence, the equilibrium solution $X^\star$ is stable.

Problem 4

Consider the nonlinear system

$$x' = x\,(1 - x/a - y)\,, \qquad y' = y\,(x - 1)\,, \tag{6}$$

where $a > 1$ is a constant.

a

Determine the nontrivial equilibrium solution $(x^\star, y^\star) \neq (0,0)$. Use the Liapunov stability theorem to prove that $(x^\star, y^\star)$ is asymptotically stable. Provide a definition of "asymptotically stable" equilibrium solution.

Hint: Use

$$V(x,y) = x - \ln x + y + C \ln y, \quad (x,y) \in O := \{x > 0, y > 0\},$$

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with an appropriately chosen C, to construct a strict Liapunov function.

Answer: The nontrivial equilibrium solution is

$$x^\star = 1, \quad y^\star = 1 - \frac{1}{a},$$

so that $(x^\star, y^\star)$ belongs to the region O. Let us define

$$L(x,y) := V(x,y) - V(x^\star, y^\star).$$

Below we justify the choice $C = \frac{1}{a} - 1 < 0$. See Figure 9. Then

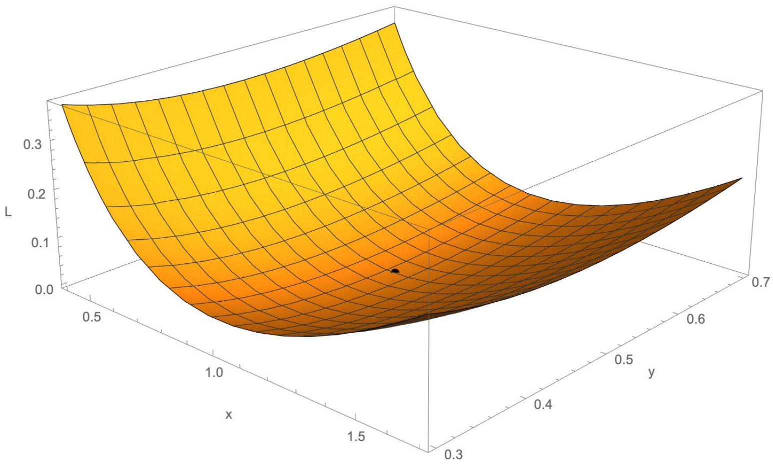

Figure 9: Problem 3–(a). Liapunov function $L(x,y)$ with $a = 2$ and thus $C = -1/2$, which implies $(x^\star, y^\star) = (1, 1/2)$.

$$L(x^\star, y^\star) = 0, \qquad L(x,y) > 0 \quad \forall (x,y) \neq (x^\star, y^\star).$$

Set $F = \big(x\,(1 - x/a - y)\,, y\,(x-1)\big)$, and let us compute

$$\begin{aligned}
&\nabla L(x,y) \cdot F(x,y) \\
&\quad = (1 - 1/x, 1 + C/y) \cdot (x\,(1 - x/a - y)\,, y\,(x-1)) \\
&\quad = -\frac{1}{a}(x-1)(x - a - aC) \\
&\quad\quad = -\frac{1}{a}(x-1)^2 \le 0 \quad (\text{if } C = 1/a - 1).
\end{aligned}$$

Hence, since (with $C = \frac{1}{a} - 1$)

$$\nabla L(x,y) \cdot F(x,y) < 0, \quad (x,y) \in O \setminus \{(x^\star, y^\star)\}\,,$$

we can use the Liapunov stability theorem to conclude that $(x^\star, y^\star)$ is asymptotically stable. We recall that "asymptotically stable" means that $X^\star$ is stable, cf. Problem 3–(f), and $\delta > 0$ can be chosen such that

$$|X_0 - X^\star| < \delta \Longrightarrow \lim_{t \to \infty} X(t) = X^\star.$$

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b

Consider the nonlinear system (6), and the nontrivial equilibrium solution $(x^\star, y^\star)$. Let $(x(t), y(t))$ be a T-periodic solution of (6). Prove that

$$\langle x\rangle_T := \frac{1}{T}\int_0^T x(t)\,dt = x^\star, \qquad \langle y\rangle_T := \frac{1}{T}\int_0^T y(t)\,dt = y^\star.$$

Answer: Using (6), we obtain

$$\frac{d}{dt}\ln x(t) = \frac{x'(t)}{x(t)} = 1 - x/a - y$$

and

$$\frac{d}{dt}\ln y(t) = \frac{y'(t)}{y(t)} = x(t) - 1.$$

Let us compute the T-averages of these equations:

$$\left\langle \frac{d}{dt}\ln x \right\rangle_T = 1 - \frac{1}{a}\langle x\rangle_T - \langle y\rangle_T$$

and

$$\left\langle \frac{d}{dt}\ln y \right\rangle_T = \langle x\rangle_T - 1.$$

By the T-periodicity of $x(t)$ and $y(t)$,

$$\left\langle \frac{d}{dt}\ln x \right\rangle_T = \frac{1}{T}\left(\ln x(T) - \ln x(0)\right) = 0$$

and

$$\left\langle \frac{d}{dt}\ln y \right\rangle_T = \frac{1}{T}\left(\ln y(T) - \ln y(0)\right) = 0.$$

Hence, $\langle x\rangle_T = 1 = x^\star$ and $\langle y\rangle_T = 1 - \frac{1}{a} = y^\star$.

THE END