UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Monday, June 14th, 2021

Examination hours: 09.00 – 13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: All aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Consider the coefficient matrix

$$A = \begin{pmatrix} 2 & -5 \\ a & -2 \end{pmatrix},$$

which depends on a parameter $a \in \mathbb{R}$. Use trace-determinant analysis to determine the phase portrait—saddle, (spiral) sink, (spiral) source or center—of the linear system of differential equations $X' = AX$.

Determine the general solution of $X' = AX$ for $a = \frac{3}{5}$.

b

Consider the nonlinear system

$$\begin{aligned} x' &= -x + x^2 + y - y^2, \\ y' &= 2x + xy. \end{aligned}$$

Determine the (four) equilibrium solutions. Use the linearization method to determine the phase portrait near each equilibrium solution.

c

Consider the nonlinear differential equation

$$x' = f_r(x) := x(x-2) + r,$$

where r is a parameter. Determine the equilibrium solutions and classify their stability (source / sink). Plot slope lines and a bifurcation diagram.

(Continued on page 2.)

Problem 2

a

Consider the nonlinear system

$$x' = -y^3, \quad y' = x^3.$$

Verify that $(0,0)$ is a non-hyperbolic equilibrium point. Plot the corresponding phase portrait. Use the Liapunov stability theorem to prove that $(0,0)$ is stable. Plot your Liapunov function.

b

Consider the nonlinear system

$$x' = -2y + yz, \quad y' = x - xz, \quad z' = xy.$$

Verify that the origin $(x, y, z) = (0, 0, 0)$ is a non-hyperbolic equilibrium point. Employ the Liapunov stability method to show that the origin is stable. Hint: Try to construct a Liapunov function of the form

$$L(x, y, z) = ax^2 + by^2 + cz^2,$$

for some suitable coefficients $a, b, c \in \mathbb{R}$.

c

Consider the second-order differential equation

$$x'' + f(x)x' + g(x) = 0, \tag{1}$$

which arises in numerous models in physics, chemistry, and biology. In its mechanical interpretation the equation models the movement of a mass subjected to a damping force $-f(x)x'$ and a restoring force $-g(x)$, where f, g are given continuously differentiable functions. In what follows, define

$$F(x) = \int_0^x f(z)\,dz, \quad G(x) = \int_0^x g(z)\,dz.$$

(i) Argue that (1) can be written as the following system of first order differential equations:

$$x' = y - F(x), \quad y' = -g(x). \tag{2}$$

Suppose $G(x) > 0$ for all $x \neq 0$. We call the quantity

$$E(t) := G(x(t)) + \frac{1}{2}y^2(t)$$

the total energy of the system at time t, which consists of potential energy $G(x(t))$ and kinetic energy $\frac{1}{2}y^2(t)$.

(ii) Suppose $g(x)F(x) > 0$ for all $x \neq 0$. Under this assumption, prove that the total energy is strictly decreasing in time t.

(Continued on page 3.)

d

Show that $(x, y) = (0, 0)$ is an asymptotically stable equilibrium solution of (2), under the same assumptions as in Problem 2c and also $g(0) = 0$.

Problem 3

a

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two continuously differentiable functions, and introduce the vector field $F(x, y) = (f(x, y), g(x, y))$. Consider the system of differential equations

$$x' = f(x, y), \quad y' = g(x, y).$$

Prove that this is a Hamiltonian system if and only if $\operatorname{div} F = 0$, where div denotes the divergence of the vector field F with respect to x, y.

b

Consider the Hamiltonian system

$$x' = H_y(x, y), \quad y' = -H_x(x, y), \tag{3}$$

where the Hamiltonian function $H = H(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable, and $H_x = \frac{\partial H}{\partial x}$ and $H_y = \frac{\partial H}{\partial y}$ denote the partial derivatives of H with respect to x and y, respectively. Suppose $(x, y) = (0, 0)$ is an equilibrium solution of (3), and that

$$H_{xx}(0, 0)H_{yy}(0, 0) - (H_{xy}(0, 0))^2 > 0. \tag{4}$$

Prove that the equilibrium solution $(0, 0)$ is a center of the linearized system.

c

Consider the nonlinear system

$$x' = y + x^2 - y^2, \quad y' = -x - 2xy.$$

Explain why this is a Hamiltonian system. Plot the phase portrait. Prove that the equilibrium solution $(x, y) = (0, 0)$ is a center of the linearized system.

THE END