UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Monday, June 14th, 2021

Examination hours: 09.00 – 13.00

This problem set consists of 8 pages.

Appendices: None

Permitted aids: All aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

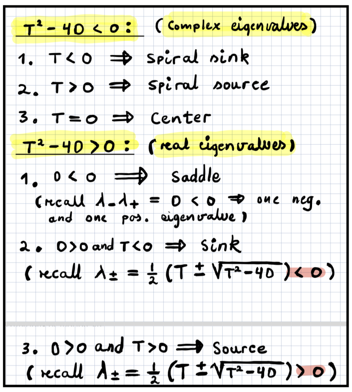
Consider the coefficient matrix

$$A = \begin{pmatrix} 2 & -5 \\ a & -2 \end{pmatrix},$$

which depends on a parameter $a \in \mathbb{R}$. Use trace-determinant analysis to determine the phase portrait—saddle, (spiral) sink, (spiral) source or center—of the linear system of differential equations $X' = AX$.

Determine the general solution of $X' = AX$ for $a = \frac{3}{5}$.

Answer: The trace is $T = 0$, the determinant is $D = -4 + 5a$, and the discriminant is $T^2 - 4D = 4 - 5a$. The matrix A has complex eigenvalues if $T^2 - 4D < 0 \iff 4 - 5a < 0 \iff a > \frac{4}{5}$. Since $T = 0$, we have in this case that the phase portrait is a center. We have real eigenvalues if $T^2 - 4D > 0 \iff 4 - 5a > 0 \iff a < \frac{4}{5}$. Since $D < 0$, in this case the phase portrait is a saddle.



(Continued on page 2.)

Finally, if $a = \frac{4}{5}$, then the determinant $D = 0$. In this case, the (repeated) eigenvalues of

$$A = \begin{pmatrix} 2 & -5 \\ \frac{4}{5} & -2 \end{pmatrix}$$

is $\lambda = 0$. Let us determine the phase portrait in this case. Adding -2 times the first differential equation to 5 times the second equation gives $-2x'(t) + 5y'(t) = 0$, so that in the xy-plane the solution curves are given by $\frac{dy}{dx} = -\frac{2}{5}$, which implies that they are straight lines given by $y = \frac{2}{5}x + C$, for any constant C.

If $a = \frac{3}{5}$, then the resulting matrix

$$A = \begin{pmatrix} 2 & -5 \\ \frac{3}{5} & -2 \end{pmatrix}$$

has eigenvalues $\lambda_{\pm} = \pm 1$. The corresponding eigenvectors are $V_- = \begin{pmatrix} \frac{5}{3} \\ 1 \end{pmatrix}$ and $V_+ = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. This gives the generalized solution

$$X(t) = c_1 e^{\lambda_- t} V_- + c_2 e^{\lambda_+ t} V_+ = \begin{pmatrix} \frac{5}{3} c_1 e^{-t} + 5 c_2 e^{t} \\ c_1 e^{-t} + c_2 e^{t} \end{pmatrix},$$

for any $c_1, c_2 \in \mathbb{R}$.

b

Consider the nonlinear system

$$x' = -x + x^2 + y - y^2,$$
$$y' = 2x + xy.$$

Determine the (four) equilibrium solutions. Use the linearization method to determine the phase portrait near each equilibrium solution.

<u>Answer:</u> The equilibrium points, i.e., the solutions of

$$-x + x^2 + y - y^2 = 0, \quad 2x + xy = 0,$$

are

$$(-2, -2), \quad (0, 0), \quad (0, 1), \quad (3, -2).$$

Set

$$F(x, y) = \left(-x + y + x^2 - y^2, 2x + xy\right).$$

The Jacobian matrix is

$$J(x, y) := DF(x, y) = \begin{pmatrix} 2x - 1 & 1 - 2y \\ y + 2 & x \end{pmatrix}.$$

<u>$(x, y) = (-2, -2)$:</u>

$$J(-2, -2) = \begin{pmatrix} -5 & -5 \\ 0 & -2 \end{pmatrix}.$$

(Continued on page 3.)

The determinant is $D = 10 > 0$, the trace is $T = -7 < 0$, and the discriminant is $T^2 - 4D = 9 > 0$. Hence $(-2, -2)$ is a sink.

$(x, y) = (0, 0)$:

$$J(0,0) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}.$$

The determinant is $D = -2 < 0$. Hence $(0, 0)$ is a saddle.

$(x, y) = (0, 1)$:

$$J(0,1) = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}.$$

The determinant is $D = 3 > 0$, the trace is $T = -1 < 0$, and the discriminant is $T^2 - 4D = -11 < 0$. Hence $(0, 1)$ is a spiral sink.

$(x, y) = (3, -2)$:

$$J(3,-2) = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}.$$

The determinant is $D = 15 > 0$, the trace is $T = 8 > 0$, and the discriminant is $T^2 - 4D = 4 > 0$. Hence $(3, -2)$ is a source.

c

Consider the nonlinear differential equation

$$x' = f_r(x) := x(x-2) + r,$$

where r is a parameter. Determine the equilibrium solutions and classify their stability (source / sink). Plot slope lines and a bifurcation diagram.

Answer: The equilibrium points satisfy $x^2 - 2x + r = 0$ and are thus

$$x = 1 \pm \sqrt{1-r};$$

we have two equilibrium points ($x = 1 \pm \sqrt{1-r}$) when $r < 1$ and one equilibrium point ($x = 1$) when $r = 1$. We compute $f_r'(x) = 2(x-1)$, and so

$$f_r'\left(1 + \sqrt{1-r}\right) = 2\sqrt{1-r} > 0,$$
$$f_r'\left(1 - \sqrt{1-r}\right) = -2\sqrt{1-r} < 0,$$

for $r < 1$. As a result, $x = 1 + \sqrt{1-r}$ is a source (unstable) and $x = 1 - \sqrt{1-r}$ is a sink (stable). Moreover, $f_r'(1) = 0$ so $x = 1$ (with $r = 1$) is a non-hyperbolic equilibrium point.

As r increases to 1, we go from two equilibrium points ($r < 1$) to one equilibrium point ($r = 1$). In other words, $r = 1$ is a saddle-node bifurcation.

Problem 2

a

Consider the nonlinear system

$$x' = -y^3, \quad y' = x^3.$$

(Continued on page 4.)

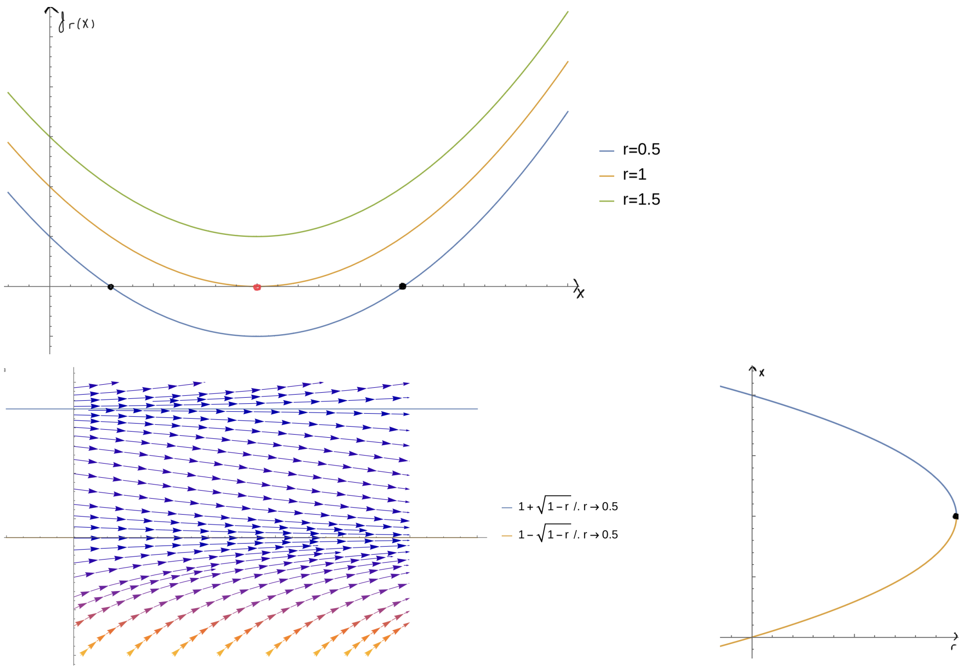


Figure 1: Problem 1c. Plot of $f_r(x)$ (top). Plot of slope lines (bottom left) for $r = \frac{1}{2}$. Bifurcation plot (bottom right)—blue ($x = 1+\sqrt{1-r}$) and yellow ($x = 1 - \sqrt{1-r}$).

Verify that $(0, 0)$ is a non-hyperbolic equilibrium point. Plot the corresponding phase portrait. Use the Liapunov stability theorem to prove that $(0, 0)$ is stable. Plot your Liapunov function.

Answer: Set $F(x, y) = \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix}$. Then $DF(x, y) = \begin{pmatrix} 0 & -3y^2 \\ 3x^2 & 0 \end{pmatrix}$, and so $DF(0, 0)$ becomes the zero matrix. This verifies that the origin is a non-hyperbolic equilibrium point. A possible Liapunov function is

$$L(x, y) = x^4 + y^4.$$

Then $L(0, 0) = 0$ and $L(x, y) > 0$ for all $(x, y) \neq (0, 0)$. Moreover,

$$\begin{aligned} \dot{L}(x, y) &= DL(x, y) \cdot F(x, y) \\ &= \left(4x^3, 4y^3\right) \cdot \left(-y^3, x^3\right) = -4x^3y^3 + 4y^3x^3 = 0. \end{aligned}$$

In other words, L is constant along solutions of the system of differential equations: $\frac{d}{dt} L(x(t), y(t)) = 0$. The Liapunov stability theorem then implies that $(0, 0)$ is stable.

b

Consider the nonlinear system

$$x' = -2y + yz, \quad y' = x - xz, \quad z' = xy.$$

Verify that the origin $(x, y, z) = (0, 0, 0)$ is a non-hyperbolic equilibrium point. Employ the Liapunov stability method to show that the origin is

(Continued on page 5.)

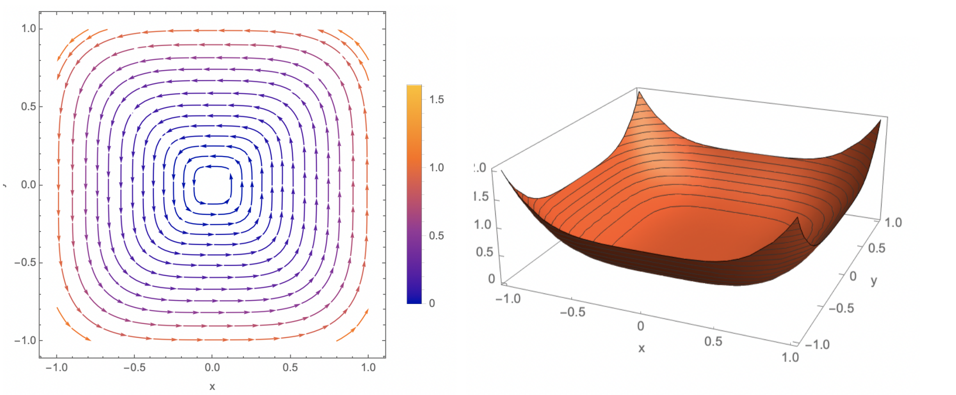


Figure 2: Problem 2a. Phase portrait (left) and Liapunov function L (right).

stable. <u>Hint</u>: Try to construct a Liapunov function of the form

$$L(x, y, z) = ax^2 + by^2 + cz^2,$$

for some suitable coefficients $a, b, c \in \mathbb{R}$.

<u>Answer:</u> Set

$$F(x, y, z) = \begin{pmatrix} -2y + yz \\ x - xz \\ xy \end{pmatrix}.$$

Clearly, $F(0, 0, 0) = 0$. Let us compute the Jacobian

$$J(x, y, z) = DF(x, y, z) = \begin{pmatrix} 0 & -2 + z & y \\ 1 - z & 0 & -x \\ y & x & 0 \end{pmatrix},$$

and so $J(0, 0, 0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The eigenvalues of this matrix, which are

$$0, \quad \pm i\sqrt{2},$$

have zero real part. Thus, $(0, 0, 0)$ is non-hyperbolic. Regarding the Liapunov function, let us compute

$$\begin{aligned} \dot{L}(x, y, z) &= DL(x, y, z) \cdot F(x, y, z) \\ &= (2ax, 2by, 2cz) \cdot (-2y + yz, x - xz, xy) \\ &= -4axy + 2axyz + 2bxy - 2bxyz + 2cxyz \\ &= (-4a + 2b)\, xy + (2a - 2b + 2c)\, xyz. \end{aligned}$$

Let us pick a, b, c such that

$$\begin{aligned} -4a + 2b = 0 &\iff b = 2a, \\ 2a - 2b + 2c = 0 &\iff c = a. \end{aligned}$$

Let us take $a = 1$, $b = 2$ and $c = 1$. Then L becomes

$$L(x, y, z) = x^2 + 2y^2 + z^2.$$

It is immediately clear that $L(0, 0, 0) = 0$ and $L(x, y, z) > 0$ for all $(x, y, z) \neq (0, 0, 0)$. Moreover, $\dot{L}(0, 0, 0) = 0$. Therefore, by the Liapunov stability theorem, $(0, 0, 0)$ is a stable equilibrium point.

(Continued on page 6.)

c

Consider the second-order differential equation

$$x'' + f(x)x' + g(x) = 0, \tag{1}$$

which arises in numerous models in physics, chemistry, and biology. In its mechanical interpretation the equation models the movement of a mass subjected to a damping force $-f(x)x'$ and a restoring force $-g(x)$, where f, g are given continuously differentiable functions. In what follows, define

$$F(x) = \int_0^x f(z)\,dz, \quad G(x) = \int_0^x g(z)\,dz.$$

(i) Argue that (1) can be written as the following system of first order differential equations:

$$x' = y - F(x), \quad y' = -g(x). \tag{2}$$

Suppose $G(x) > 0$ for all $x \neq 0$. We call the quantity

$$E(t) := G(x(t)) + \frac{1}{2}y^2(t)$$

the total energy of the system at time t, which consists of potential energy $G(x(t))$ and kinetic energy $\frac{1}{2}y^2(t)$.

(ii) Suppose $g(x)F(x) > 0$ for all $x \neq 0$. Under this assumption, prove that the total energy is strictly decreasing in time t.

Answer: (i) Let $x(t)$ and $y(t)$ solve (2). Using the chain rule, differentiating x' gives

$$x'' = y' - F'(x)x' = -g(x) - f(x)x',$$

which shows that x solves (1).

(ii) Multiply the first equation in (2) by $g(x)$ and the second equation by y. Adding the resulting equations supplies

$$g(x)x' + yy' = g(x)y - g(x)F(x) - yg(x) = -g(x)F(x).$$

By the chain rule (applied to the left-hand side) and the definition of G,

$$\frac{d}{dt}\left(G(x(t)) + \frac{1}{2}y^2(t)\right) = -g(x(t))F(x(t)).$$

In other words,

$$\frac{d}{dt}E(t) = -g(x(t))F(x(t)) < 0,$$

by our assumption. This shows that the total energy $E(t)$ is strictly decreasing in t.

(Continued on page 7.)

d

Show that $(x,y) = (0,0)$ is an asymptotically stable equilibrium solution of (2), under the same assumptions as in Problem 2c and also $g(0) = 0$.

Answer: Making use of the fact that $F(0) = g(0) = 0$, we see that $(x,y) = (0,0)$ is an equilibrium solution of (2). The simplest approach to studying the stability of $(0,0)$ consists in applying the Liapunov stability theorem. Motivated by Problem 2c, set

$$L(x,y) = G(x) + \frac{1}{2}y^2.$$

Then $L(0,0) = 0$ and $L(x,y) > 0$ for all $(x,y) \neq 0$, since $G(x) > 0\ \forall x \neq 0$. Besides,

$$\begin{aligned} \dot{L}(x,y) &= DL(x,y) \cdot (y - F(x), -g(x)) \\ &= (g(x), y) \cdot (y - F(x), -g(x)) \\ &= g(x)y - g(x)F(x) - yg(x) = -g(x)F(x) < 0, \end{aligned}$$

as long as $x \neq 0$, since $g(x)F(x)$ is assumed to be strictly positive for $x \neq 0$. Hence, L is a Liapunov function and the claim follows from the Liapunov stability theorem.

Problem 3

a

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two continuously differentiable functions, and introduce the vector field $F(x,y) = (f(x,y), g(x,y))$. Consider the system of differential equations

$$x' = f(x,y), \quad y' = g(x,y).$$

Prove that this is a Hamiltonian system if and only if $\operatorname{div} F = 0$, where div denotes the divergence of the vector field F with respect to x, y.

Answer: The system of differential equations is a Hamiltonian system $\iff$

$$x' = H_y(x,y), \quad y' = -H_x(x,y),$$

for some function $H = H(x,y) : \mathbb{R}^2 \to \mathbb{R}$. In other words, H must satisfy the equations $H_y = f$ and $H_x = -g$. Since $H_{yx} = H_{xy}$, these equations imply that $f_x = -g_y$ or $\operatorname{div} F = f_x + g_y = 0$.

b

Consider the Hamiltonian system

$$x' = H_y(x,y), \quad y' = -H_x(x,y), \tag{3}$$

where the Hamiltonian function $H = H(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable, and $H_x = \frac{\partial H}{\partial x}$ and $H_y = \frac{\partial H}{\partial y}$ denote the partial derivatives

(Continued on page 8.)

of H with respect to x and y, respectively. Suppose $(x, y) = (0, 0)$ is an equilibrium solution of (3), and that

$$H_{xx}(0,0)H_{yy}(0,0) - (H_{xy}(0,0))^2 > 0. \tag{4}$$

Prove that the equilibrium solution $(0, 0)$ is a center of the linearized system.

Answer: With $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $F(X) = \begin{pmatrix} H_y \\ -H_x \end{pmatrix}$, we can write (3) in the form

$$X' = F(X).$$

The linearized system is

$$X' = AX, \quad A = DF(0).$$

We compute

$$A = \begin{pmatrix} H_{yx}(0,0) & H_{yy}(0,0) \\ -H_{xx}(0,0) & -H_{xy}(0,0) \end{pmatrix}.$$

Since $H_{yx} = H_{xy}$, the trace is $T = 0$. The determinant is

$$D = -(H_{xy}(0,0))^2 + H_{xx}(0,0)H_{yy}(0,0).$$

Given (4), it follows that $D > 0$. Finally, the discriminant is

$$T^2 - 4D = -4D < 0.$$

Hence $(0, 0)$ is a center.

c

Consider the nonlinear system

$$x' = y + x^2 - y^2, \quad y' = -x - 2xy.$$

Explain why this is a Hamiltonian system. Plot the phase portrait. Prove that the equilibrium solution $(x, y) = (0, 0)$ is a center of the linearized system.

Answer: Set $F(x, y) = \big(y + x^2 - y^2, -x - 2xy\big)$, and compute

$$\operatorname{div} F(x, y) = 2x - 2x = 0.$$

The Hamiltonian structure then follows from Problem 3a. The Hamiltonian function is determined by the equations

$$H_y = y + x^2 - y^2, \quad -H_x = -x - 2xy.$$

A corresponding Hamiltonian function H is

$$H(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 + x^2 y.$$

In view of Problem 3b, we compute

$$\begin{aligned} &H_{xx} = 1 + 2y, \quad H_{yy} = 1 - 2y, \quad H_{xy} = 2x \\ \implies \quad &H_{xx}H_{yy} - (H_{xy})^2 = 1 - 4x^2 - 4y^2; \end{aligned}$$

therefore

$$H_{xx}(0,0)H_{yy}(0,0) - (H_{xy}(0,0))^2 = 1 > 0.$$

This proves that the origin is a center, thanks to Problem 3b.

THE END

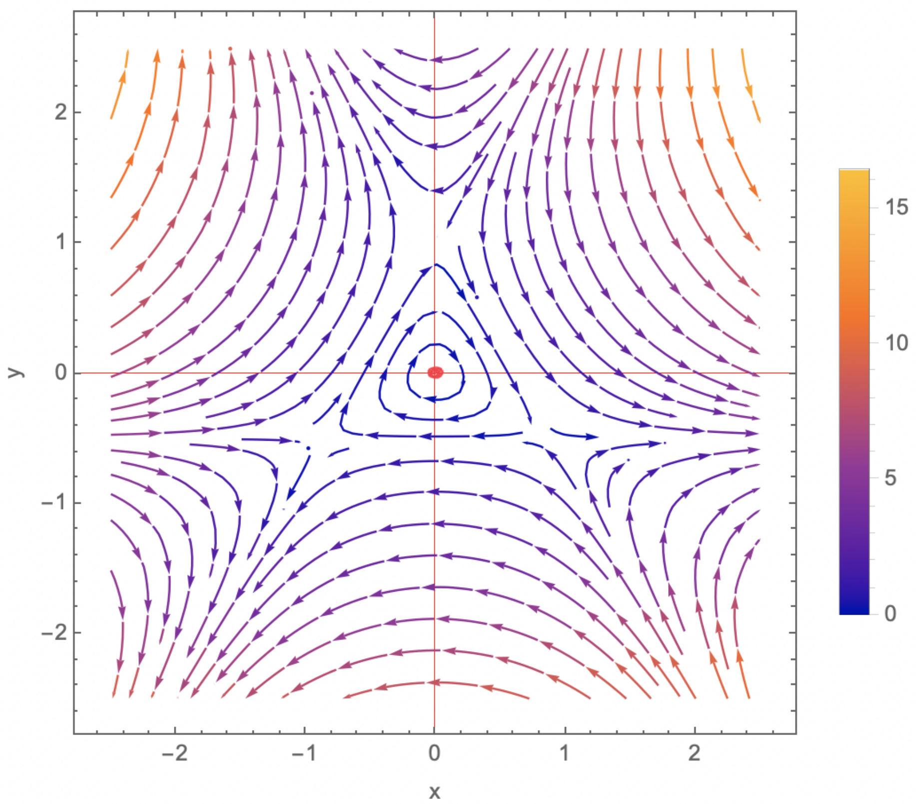


Figure 3: Problem 3c. Phase portrait.

(Continued on page 10.)