UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: 15 June 2022

Examination hours: 15:00 – 19:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- You can get a total of 110 points. The point distribution is specified for each problem.
- All answers must be justified.

Problem 1 (15p)

Consider the system

$$\begin{cases} \dot{x} = -(x-1)^3 - (x-y)^3 \\ \dot{y} = (x-y)^3 \end{cases} \tag{1}$$

1a (3p)

Show that $u^* := (1,1)$ is the only fixed point.

1b (5p)

Find the linearized system for u^*. Does the linearization tell us anything about the stability of the fixed point? Explain why, or why not.

1c (7p)

Show that u^* is an asymptotically stable fixed point.

Hint: Show first that the system is a gradient system.

Problem 2 (10p)

Compute the matrix exponential of $A := \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix}$.

(Continued on page 2.)

Problem 3 (10p)

Consider the explicit Euler (forward Euler) method with step length $h > 0$, applied to the system

$$\dot{u} = Au, \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ -1000 & -1001 \end{pmatrix}. \tag{2}$$

Find the largest number $h_0 > 0$ such that this method is linearly stable for any step length $h \in (0, h_0)$.

Hint: You may use what you have learnt in class about the stability region and stability function of the explicit Euler method.

Problem 4 (45p)

Consider the system

$$\begin{cases} \dot{x} = \alpha x - y - x^3 \\ \dot{y} = x + \alpha y - y^3 \end{cases} \tag{3}$$

for some parameter α. It can be shown that if $\alpha < 2$ then the only fixed point of (1) is the origin (you don't need to show this).

4a (5p)

For some value of $\alpha < 2$ of your choice, draw the nullclines of (3) and indicate the direction of the velocity field.

4b (5p)

Show that sets of the form $A_R \coloneqq \{(x, y) : x^2 + y^2 < R\}$ are forward invariant whenever $R > 0$ is large enough.

4c (10p)

Show that (3) has a unique solution defined for all $t \geqslant 0$ for any initial data $(x(0), y(0))$.

4d (10p)

Determine the type of stability of $(0, 0)$ (i.e., Lyapunov stable, unstable, asymptotically stable, etc.) for all values of $\alpha < 2$.

4e (10p)

Show that the system has a periodic orbit when $\alpha \in (0, 2)$.

4f (5p)

Draw a bifurcation diagram for (3). What sort of bifurcation does (3) undergo, and for what value of α?

(Continued on page 3.)

Problem 5 (30p)

Consider the Lotka–Volterra model

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y), \end{cases} \tag{4}$$

which is a model for the number of individuals in two species in an ecosystem. In particular, we require $x, y \geqslant 0$. It is easy to check that

$$p_0 := (0,0), \quad p_1 := (1,1), \quad p_2 := (0,2) \quad \text{and} \quad p_3 := (3,0)$$

are fixed points for (4) (you don't need to check this).

5a (3p)

Are the species predators or prey?

5b (5p)

Find the nullclines of (4) and use them to draw a (rough) phase portrait. Be sure to indicate the fixed points.

5c (8p)

It can be shown that p_2 and p_3 are asymptotically stable (you don't need to show this). Show that p_0 is repelling (i.e., asymptotically stable backwards in time), and that p_1 is unstable (i.e., not Lyapunov stable).

5d (14p)

What does the stable manifold theorem say about the fixed point p_1? Use this information to draw a new, more detailed phase portrait.