UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: 15 June 2022

Examination hours: 15:00 – 19:00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- You can get a total of 110 points. The point distribution is specified for each problem.
- All answers must be justified.

Problem 1 (15p)

Consider the system

$$\begin{cases} \dot{x} = -(x-1)^3 - (x-y)^3 \\ \dot{y} = (x-y)^3 \end{cases} \tag{1}$$

1a (3p)

Show that $u^* := (1,1)$ is the only fixed point.

1b (5p)

Find the linearized system for u^*. Does the linearization tell us anything about the stability of the fixed point? Explain why, or why not.

1c (7p)

Show that u^* is an asymptotically stable fixed point.

Hint: Show first that the system is a gradient system.

Solution:

1a

It is clear that u^* is a fixed point. At any fixed point we need $(x-y)^3 = 0$, that is, $x = y$, and hence also $(x-1)^3 = 0$, so $x = 1$, whence $y = 1$.

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1b

The linearized system around u^* is $\dot{v} = Av$, where $A = \nabla F(u^*) = 0$. The eigenvalues of A are 0, so u^* is not hyperbolic. We can therefore not use the linearized system to learn about the stability of the fixed point.

1c

We claim that $\dot{u} = -\nabla G(u)$ for some $G\colon \mathbb{R}^2 \to \mathbb{R}$. Indeed, by the equation for $\dot{y}$, G must contain a term $\frac{1}{4}(x-y)^4$, and by the equation for $\dot{x}$, G must contain the terms $\frac{1}{4}(x-1)^4$ and $\frac{1}{4}(x-y)^4$. The function $G(x,y) := \frac{1}{4}(x-1)^4 + \frac{1}{4}(x-y)^4$ fits both these needs.

This function G has u^* as its unique, global minimum. Hence, G is a Lyapunov function for u^* on all of $\mathbb{R}^2$, so by the theory of Lyapunov functions, u^* is asymptotically stable (in fact, globally attracting).

Problem 2 (10p)

Compute the matrix exponential of $A := \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix}$.

Solution: A is upper triangular so the eigenvalues are the diagonal entries: $\lambda_1 = -1$ and $\lambda_2 = 2$. The corresponding eigenvectors are $r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Denote $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$ and $R = (r_1\, r_2)$. Then $A = R\Lambda R^{-1}$, where $R^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = R$. We get

$$e^A = Re^\Lambda R^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{-1} & e^{-1} - e^2 \\ 0 & e^2 \end{pmatrix}.$$

Problem 3 (10p)

Consider the explicit Euler (forward Euler) method with step length $h > 0$, applied to the system

$$\dot{u} = Au, \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ -1000 & -1001 \end{pmatrix}. \tag{2}$$

Find the largest number $h_0 > 0$ such that this method is linearly stable for any step length $h \in (0, h_0)$.

Hint: You may use what you have learnt in class about the stability region and stability function of the explicit Euler method.

Solution: The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -1000$. In class we have found that the stability region of the explicit Euler method is $\mathcal{D} = \{z \in \mathbb{C} : |z+1| \leqslant 1\}$. Since we need $\lambda_k h \in \mathcal{D}$ for $k = 1, 2$, and the eigenvalues are real, we need $\lambda_k h \in \mathcal{D} \cap \mathbb{R} = [-2, 0]$, so we need

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$h \leqslant 2/|\lambda_k|$ for $k = 1, 2$. This yields the value $h_0 = 2/1000 = 1/500$.

Problem 4 (45p)

Consider the system

$$\begin{cases} \dot{x} = \alpha x - y - x^3 \\ \dot{y} = x + \alpha y - y^3 \end{cases} \tag{3}$$

for some parameter α. It can be shown that if $\alpha < 2$ then the only fixed point of (1) is the origin (you don't need to show this).

4a (5p)

For some value of $\alpha < 2$ of your choice, draw the nullclines of (3) and indicate the direction of the velocity field.

4b (5p)

Show that sets of the form $A_R \coloneqq \{(x, y) : x^2+y^2 < R\}$ are forward invariant whenever $R > 0$ is large enough.

4c (10p)

Show that (3) has a unique solution defined for all $t \geqslant 0$ for any initial data $(x(0), y(0))$.

4d (10p)

Determine the type of stability of $(0, 0)$ (i.e., Lyapunov stable, unstable, asymptotically stable, etc.) for all values of $\alpha < 2$.

4e (10p)

Show that the system has a periodic orbit when $\alpha \in (0, 2)$.

4f (5p)

Draw a bifurcation diagram for (3). What sort of bifurcation does (3) undergo, and for what value of α?

Solution:

4a

...

4b

We study the evolution of $L(x, y) \coloneqq x^2 + y^2$:

$$\dot{L} = 2x(\alpha x - y - x^3) + 2y(\alpha x + y - y^3) = 2\big(\alpha(x^2 + y^2) - x^4 - y^4\big),$$

(Continued on page 4.)

which is negative whenever $x^2 + y^2$ is large enough. (A proof of this goes as follows: By the elementary inequality $2ab \leqslant a^2 + b^2$ we get

$$\dot{L} \leqslant 2\alpha(x^2 + y^2) - x^4 - y^4 - 2x^2y^2 = 2\alpha(x^2 + y^2) - (x^2 + y^2)^2,$$

which is strictly negative when $x^2 + y^2 > 2\alpha$.) Hence, any solution on the boundary of ∂A_R, for R large enough, will move into A_R. It follows that A_R is forward invariant.

4c

The velocity field in (3) is locally Lipschitz continuous since it is C^1 (in fact, C^∞). Therefore, it follows from the Cauchy–Lipschitz theory that there exists a unique solution defined for all $t \in (-\tau, \tau)$ for some $\tau > 0$. If $R > 0$ is large enough that $(x(0), y(0)) \in A_R$ and A_R is forward invariant, then the solution remains bounded as t increases. It follows that the solution exists for all $t \geqslant 0$.

4d

The linearization at $(0, 0)$ is

$$\dot{v} = Av, \qquad \text{where } A = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}.$$

The eigenvalues of A are $\alpha \pm i$. The real part of both eigenvalues is α, so the fixed point is repelling for $\alpha > 0$ and attracting when $\alpha < 0$. When $\alpha = 0$ we can use the estimate in problem 4b to see that $\dot{L} = -2(x^4+y^4)$, so L is a Lyapunov function for $(0, 0)$, whence the origin is attracting.

Alternative solution: From the estimate in problem 4b, we see that

$$\dot{L} \geqslant 2\alpha(x^2 + y^2) - 2(x^4 + 2x^2y^2 + y^4) = 2r^2(\alpha - r^2)$$

where $r = \|(x, y)\|$.

- When $\alpha > 0$, this quantity is strictly positive in the disc $U := B_{r_0}(0)$ (where $r_0 = \sqrt{\alpha}$), apart from at $(x, y) = (0, 0)$. Moreover, $(0, 0)$ is a strict minimum of L. It follows that L is a Lyapunov function for (3) but *backwards in time.* Hence, $(0, 0)$ is repelling, i.e. asymptotically stable backwards in time.
- When $\alpha \leqslant 0$ this quantity is strictly negative everywhere apart from at $(0, 0)$. By the same reasoning, it follows that $(0, 0)$ is (globally) attracting.

4e

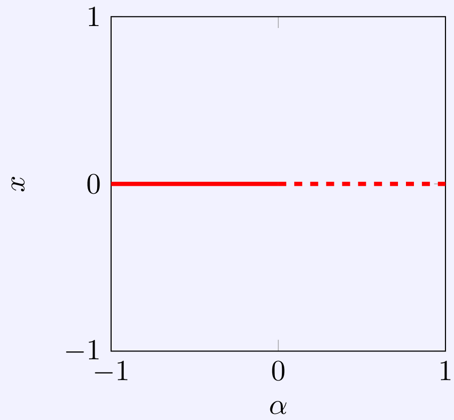
Let $R > 0$ be such that A_R is forward invariant. If $u_0 \in A_R$ is any point apart from the origin, then $\Gamma_+(u_0)$ is bounded (since A_R is bounded),

(Continued on page 5.)

so by the Poincaré–Bendixson theorem, $\omega(u_0)$ either contains a fixed point or a periodic orbit. But the only fixed point in A_R is 0, which is repelling, so $\omega(u_0)$ must be a periodic orbit. We conclude that there exists a periodic orbit.

4f

The system undergoes a Hopf bifurcation at $\alpha = 0$ near $(0,0)$. Indeed, for $\alpha < 0$ the fixed point is globally attracting, so there are no periodic orbits, while for $\alpha > 0$ there is a periodic orbit.



Problem 5 (30p)

Consider the Lotka–Volterra model

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y), \end{cases} \tag{4}$$

which is a model for the number of individuals in two species in an ecosystem. In particular, we require $x, y \geqslant 0$. It is easy to check that

$$p_0 := (0,0), \quad p_1 := (1,1), \quad p_2 := (0,2) \quad \text{and} \quad p_3 := (3,0)$$

are fixed points for (4) (you don't need to check this).

5a (3p)

Are the species predators or prey?

5b (5p)

Find the nullclines of (4) and use them to draw a (rough) phase portrait. Be sure to indicate the fixed points.

5c (8p)

It can be shown that p_2 and p_3 are asymptotically stable (you don't need to show this). Show that p_0 is repelling (i.e., asymptotically stable backwards in time), and that p_1 is unstable (i.e., not Lyapunov stable).

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5d (14p)

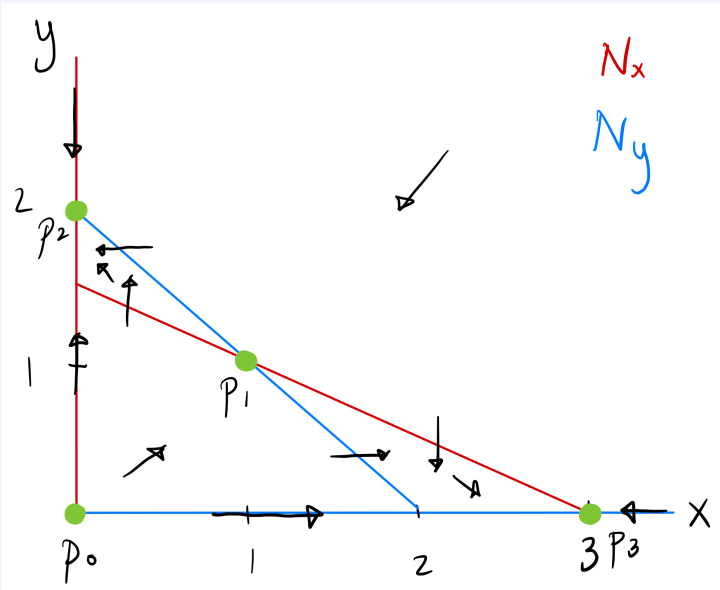
What does the stable manifold theorem say about the fixed point p_1? Use this information to draw a new, more detailed phase portrait.

Solution:

5a

Both species are prey animals, since neither specie will benefit from the abundance of the other specie.

5b



The figure above shows the nullclines and arrows indicating the direction of the velocity field.

5c

The Jacobian of the velocity field is

$$\nabla F(x,y) = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}.$$

Hence,

$$\nabla F(p_0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \nabla F(p_1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}.$$

Hence, the eigenvalues of p_0 are $3, 2$, which are positive, so p_0 is repelling. The eigenvalues of p_1 are $\lambda_\pm - 1 \pm \sqrt{2}$, which have opposing signs, so p_1 is unstable.

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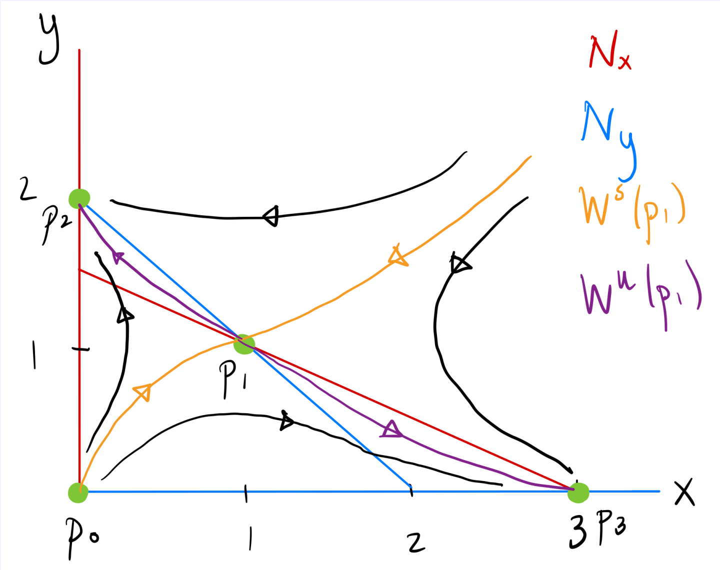
5d

The eigenvectors of $\nabla F(p_1)$ corresponding to $\lambda_\pm = -1 \pm \sqrt{2}$ are $r_\pm = \begin{pmatrix} \mp\sqrt{2} \\ 1 \end{pmatrix}$. According to the stable manifold theorem, there are two branches of the stable manifold emanating from p_1 in the directions $\pm r_-$, and two branches of the unstable manifold emanating from p_1 in the directions $\pm r_+$. According to the phase portrait in problem 5b, the two branches of the stable manifold must come from p_0 and (∞, ∞), and the two branches of the unstable manifold must approach p_2 and p_3 as $t \to \infty$.



The above figure shows the nullclines, along with the stable manifold (orange) and the unstable manifold (purple) of p_1, as well as four other orbits.