Deadline: Thursday 2. April 2020

MANDATORY ASSIGNMENT MAT3440—SPRING 2020

INFORMATION

All mandatory assignments must be uploaded via Canvas.

- *The assignment must be submitted as a single PDF file.*
- *Scanned pages must be clearly legible.*
- *The submission must contain your name, course and assignment number.*

If these requirements are not met, the assignment will not be evaluated. Read the information about mandatory assignments carefully: http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html

To have a passing grade you must have satisfactory answers to at least 50% of the questions and have attempted to solve all of them.

PROBLEM 1

a)

Consider the nonlinear differential equation

$$x' = f_r(x) := rx - x^3,$$

which depends on a parameter $r \in \mathbb{R}$. Determine the equilibrium solutions and classify their stability (source / sink). Draw slope fields, phase lines, and a bifurcation diagram.

b)

Consider the nonlinear differential equation

$$x' = g_b(x) := x - x^3 + b,$$

which depends on a parameter $b \in \mathbb{R}$. Explain what happen to the equilibrium solutions when we change the parameter b? Sketch a bifurcation diagram.

c)

Consider the following differential equation with 2π-periodic forcing term:

$$x' = f(t, x) = -x + 2\cos t.$$

Compute the Poincaré map $p(x_0)$.

d)

Determine the fixed point x_f of the Poincare map, and the corresponding 2π-periodic solution $x_f(t)$ of $x' = f(t, x), x(0) = x_f$. What happens with any other solution as $t \to \infty$.

PROBLEM 2

a)

Find the general solution and draw the phase portrait for the linear system

$$x' = -y, \quad y' = x.$$

Determine the solution that satisfies $x(0) = x_0, y(0) = y_0$.

b)

Find the general solution and draw the phase portrait for the linear system

$$x' = -x + y, \quad y' = -y.$$

Determine the solution that satisfies $x(0) = x_0, y(0) = y_0$. What happens to the solution $\big(x(t), y(t)\big)$ as $t \to \infty$?

c)

Compute $\exp(A)$, where $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$.

<u>Hint</u>: Introducing the complex number $\lambda = a + ib$ with real part Re$(z) = a$ and imaginary part Im$(z) = b$, you can use that

$$A^k = \begin{pmatrix} \text{Re}\left(\lambda^k\right) & -\text{Im}\left(\lambda^k\right) \\ \text{Im}\left(\lambda^k\right) & \text{Re}\left(\lambda^k\right) \end{pmatrix}, \qquad k = 1,2,\ldots,$$

where Re $\left(\lambda^k\right)$, Im $\left(\lambda^k\right)$ denote respectively the real and imaginary parts of the complex number λ^k.

d)

Consider the matrix $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, with $a, b \in \mathbb{R}$. Prove that

$$\exp(A) = e^a \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

e)

Use the matrix exponential to solve the initial value problem

$$X' = AX, \quad X(0) = X_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}.$$

Sketch the solution curve in the phase plane.

f)

Solve the nonautonomous linear system

$$X' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} X + \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}, \quad X(0) = X_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

PROBLEM 3

Consider the nonlinear differential equation

(2) $$x' = f(x) = x^2, \quad x(0) = 1.$$

a)

Determine the solution $x(t)$ of (2) for $t \in (-\infty, 1)$. What happens with $x(t)$ as $t \uparrow 1$?

b)

Compute the Picard approximations $u_0(t), u_1(t), u_2(t), u_3(t)$.

c)

Consider the (scalar) nonlinear differential equation $x' = f(t, x)$ with initial data $x(0) = x_0$, for $t \in [-a, a]$, $a > 0$, where $f(t, x)$ is a function that is continuous in $t \in [-a, a]$ and continuously differentiable in $x \in [-\rho, \rho]$, $\rho > 0$. In fact, let us assume

$$\left| f(t, x) - f(t, y) \right| \leq K |x - y|, \quad \forall x, y \in [-\rho, \rho],$$

uniformly in $t \in [-a, a]$, and

$$\left| f(t, x) \right| \leq M, \quad \forall t \in [-a, a], \forall x \in [-\rho, \rho],$$

for some positive constants K, M. In what follows, fix $a > 0$ such that

$$aM < \rho, \quad aK < 1.$$

Prove that the Picard iterations $\left\{u_k(t)\right\}_{k=0}^{\infty}$ defined by $u_0(t) = x_0$ and

$$u_{k+1}(t) = x_0 + \int_0^t f(s, u_k(s))\, ds, \quad k = 0,1,2,\ldots,$$

converge uniformly to a limit function $u(t)$ by establishing the following two estimates:

(i) $u_k(t) \in \left[x_0 - \rho, x_0 + \rho\right]$, that is $|u_k(t) - x_0| \leq \rho$, for all $t \in [-a, a]$;

(ii) $\left|u_i(t) - u_j(t)\right| \xrightarrow{i,j\uparrow\infty} 0$, uniformly in $t \in [-a, a]$.