WHAT IS AN ODE?

Let $x : \mathbb{R} \to \mathbb{R}^n$ be given, $x = x(t)$.

We often think of t as time.

<u>Notation</u>

We denote the derivative of x at t by

$$\dot{x}(t) = x'(t) = \frac{dx}{dt}(t) = \lim_{h \to 0} \frac{1}{h}\Big(x(t+h) - x(t)\Big)$$

- If $x(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$ then $\dot{x}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^n(t) \end{pmatrix}$.
- We often hide the dependence on t: $x = x(t)$, $\dot{x} = \dot{x}(t)$, ...
- The higher derivatives of x are $\ddot{x}$, $\dddot{x}$, ...

An <u>ordinary differential equation</u> (ODE) with unknown $x = x(t)$ is an equation involving x and its derivatives:

$$\boxed{\dot{x} = 3x}$$

$$\boxed{\ddot{x} - 2\dot{x} + 3t\dot{x} = 4e^{t}}$$

$$\boxed{2\dddot{x} + x^{2} = t\cos(\dot{x})}$$

- <u>Solving</u> the ODE means finding a function $x = x(t)$ which satisfies the ODE for <u>every</u> t.
- The function x is the <u>solution</u> of the ODE.

An ODE is <u>linear</u> if x (and $\dot{x}, \ddot{x}, \ldots$) appear <u>linearly</u> in the equation.

$$2\dot{x} + 3x = 5t$$
$$\ddot{x} - 4x = 3\dot{x}t^2$$

linear

$$\ddot{x} + \frac{1}{2}x^2 = -\dot{x}$$
$$\dot{x} + \sin(x) - 3t = 0$$

nonlinear

The ODE $\dot{x} = f(x,t)$ is linear if f is linear in x:

$$f(\alpha x + \beta y, t) = \alpha f(x,t) + \beta f(y,t) \qquad \forall\ \alpha, \beta, t \in \mathbb{R}$$
$$\forall\ x, y \in \mathbb{R}^n$$

The **order** of the ODE is the highest number of derivatives occuring in the ODE.

$$\boxed{\dot{x} = 3x}$$ first-order

$$\boxed{\ddot{x} - 2\dot{x} + 3t\dot{x} = 4e^{t}}$$ second-order

$$\boxed{2\dddot{x} + x^2 = t\cos(\dot{x})}$$ third-order

It is common to put the term with the highest derivative first in the equation.

A system of ODEs is a set of n ODEs for n unknowns:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$\begin{cases} \dot{x}_1 + 2x_2^2 = t \\ \dot{x}_2 = 3x_1x_3 \\ \ddot{x}_3 - 5x_2 + x_1x_2 = 0 \end{cases}$$

In order to determine a <u>unique</u> solution, we need to provide <u>initial data</u> (or <u>boundary data</u>):

$$\boxed{x(0) = 5}$$

$$\boxed{x(a) = 2, \quad x(b) = 0}$$

$$\boxed{x(3) = 0, \quad x'(5) = 2}$$

- As a rule of thumb, we need as many initial data as the order of the equation.
- An <u>initial value problem</u> is an ODE together with initial data.

We can always reduce the order to 1

Example: $3\dddot{x} - 5\ddot{x} + 2\dot{x} + 8x = 15$

Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$. Then

$$\begin{cases} 3\dot{x}_3 - 5x_3 + 2x_2 + 8x_1 = 15 \\ \dot{x}_2 = x_3 \\ \dot{x}_1 = x_2 \end{cases}$$

which is a system of first-order ODEs

An <u>autonomous</u> ODE is an ODE where t does not appear explicitly.

$$\boxed{\begin{array}{c}\dot{x} + 3x^2 + 5 = 0 \\ \text{autonomous}\end{array}}$$

$$\boxed{\begin{array}{c}\dot{x} + 3x^2 + 5t^2 = 0 \\ \text{non-autonomous}\end{array}}$$

We can always turn a non-aut. ODE into an aut. ODE:

Consider $\dot{x}(t) = f(x(t), t)$, $\quad x : \mathbb{R} \to \mathbb{R}^n$.

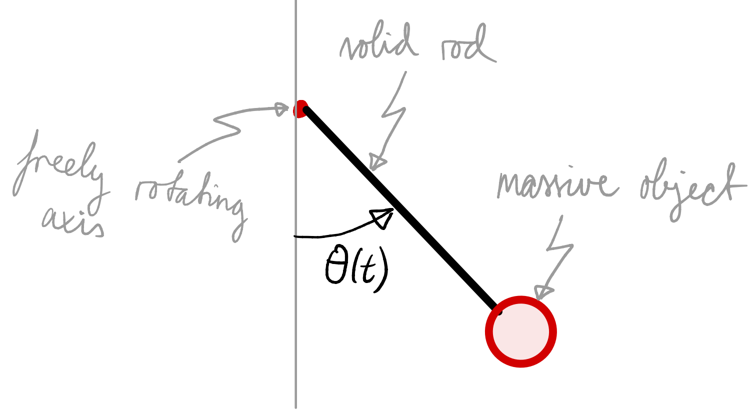
Let $y(t) = \begin{pmatrix} x(t) \\ t \end{pmatrix} \in \mathbb{R}^{n+1}$. Then

$$\dot{y}(t) = \begin{pmatrix} \dot{x}(t) \\ 1 \end{pmatrix} = \begin{pmatrix} f(x(t), t) \\ 1 \end{pmatrix} = \begin{pmatrix} f(y(t)) \\ 1 \end{pmatrix} = g(y(t))$$

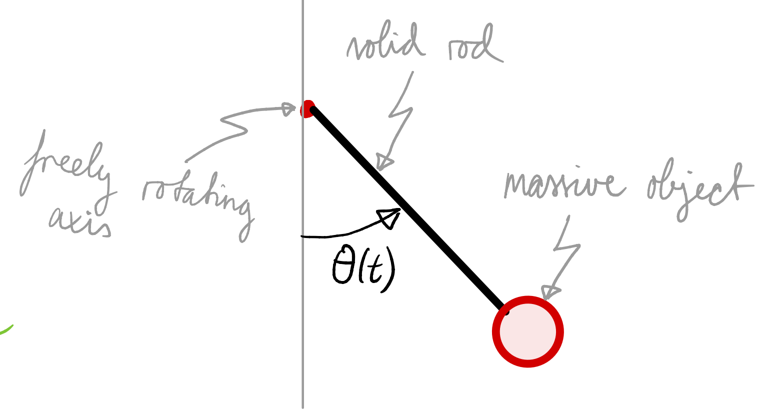
which is autonomous.

Example: The pendulum

- $\theta(t)$: angle between rod and vertical direction
- l : length of rod
- m : mass of object



Example: The pendulum

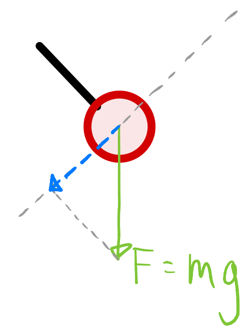


- Newton's 2nd law:

  force = mass · acceleration

- Acceleration of the object $= l\ddot{\theta}$

- Force due to gravity $= -mg\sin\theta$



Thus: $-mg\sin\theta = ml\ddot{\theta}$, or $\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$

QUESTIONS ?

COMMENTS ?