SOLVING SOME FIRST-ORDER ODEs

Consider the <u>initial value problem</u>

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(0) = x_0 \end{cases} \qquad (t \in \mathbb{R} \quad \text{or} \quad t \in (0, \infty))$$

for some $x_0 \in \mathbb{R}$ (or $x_0 \in \mathbb{R}^n$).

<u>Most ODEs cannot be solved explicitly</u>!

- There might <u>exist</u> a solution, but in most cases it does not have a simple formula.
- <u>Separable</u> ODEs do have a solution formula.

A <u>separable</u> ODE is a scalar, first-order ODE of the form

$$\boxed{\dot{x} = g(x)h(t)}$$

for $g, h : \mathbb{R} \longrightarrow \mathbb{R}$.

$$\boxed{\dot{x} = g(x)\,h(t)}$$

Divide by $g(x)$:

$$\frac{\dot{x}}{g(x)} = h(t) \implies \int_0^t \frac{\dot{x}(s)}{g(x(s))}\,ds = \underbrace{\int_0^t h(s)\,ds}_{=H(t)}$$

$$\implies H(t) = \int_0^t \frac{\dot{x}(s)}{g(x(s))}\,ds \overset{(y = x(s))}{=} \int_{x(0)}^{x(t)} \frac{1}{g(y)}\,dy = G(x(t)) - G(x(0))$$

where $G(x) = \int_0^x \frac{1}{g(y)}\,dy$. Set $x(0) = x_0$ and solve for $x(t)$:

$$\boxed{x(t) = G^{-1}\big(H(t) + G(x_0)\big)}$$

$$\boxed{\dot{x} = g(x)h(t)}$$

The only thing that could go wrong is that $g(x) = 0$.

If there is some t^* where $g(x(t^*)) = 0$, we set

$$x(t) = x(t^*) \quad \text{for all } t > t^*.$$

Then $\dot{x}(t) = 0 \ \forall \ t > t^*$, and $g(x(t))h(t) = 0 \ \forall \ t > t^*$,

so x solves the ODE.

$\boxed{\underline{\text{Example}}: \quad \dot{x} = \lambda x \quad \text{for some } \lambda \in \mathbb{R}}$

- If $x_0 = 0$ then $x(t) = 0 \ \forall t$ will be a solution.
- If $x_0 \neq 0$, divide by x and integrate:

$$\lambda = \frac{\dot{x}}{x} \quad \Longrightarrow \quad \lambda t = \int_0^t \frac{\dot{x}(s)}{x(s)}\, ds = \int_{x_0}^{x(t)} \frac{1}{y}\, dy = \log(x(t)) - \log(x_0).$$

Solve for $x(t)$ to get

$$\boxed{x(t) = x_0 e^{\lambda t}}$$

To be sure we can check: $\dot{x}(t) = \lambda x_0 e^{\lambda t} = \lambda x(t)$ for all t,

and $x(0) = x_0 e^{\lambda \cdot 0} = x_0$.

$\boxed{\text{Example: } \quad 2x^2 t - \dot{x} = -2t}$

Rewrite the ODE as $\dot{x} = 2t(1+x^2)$. Divide by $1+x^2$:

$$\frac{\dot{x}}{1+x^2} = 2t \implies t^2 = \int_0^t \frac{\dot{x}(s)}{1+x(s)^2}\,ds = \int_{x_0}^{x(t)} \frac{1}{1+y^2}\,dy$$

$$= \arctan(x(t)) - \arctan(x_0).$$

Solve for $x(t)$ to get

$$\boxed{x(t) = \tan\left(t^2 + \arctan(x_0)\right)}$$

QUESTIONS?

COMMENTS?