SOLVING SOME SECOND-ORDER ODEs

- As with first-order ODEs, second-order ODEs are in general impossible to solve exactly.

- We will solve linear, homogeneous second-order ODEs with constant coefficients,

$$a\ddot{x} + b\dot{x} + cx = 0$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

- After dividing by a we may assume $a = 1$:

$$\ddot{x} + b\dot{x} + cx = 0.$$

- We supply initial data: $x(0) = x_0$, $\dot{x}(0) = v_0$ (for $x_0, v_0 \in \mathbb{R}$).

Idea: Make the *ansatz* $x(t) = e^{\lambda t}$ for some λ.

Then $\ddot{x} + b\dot{x} + cx = e^{\lambda t}(\lambda^2 + b\lambda + c) \overset{!}{=} 0$, so we get the *characteristic equation* $\lambda^2 + b\lambda + c = 0$, with roots

$$\lambda_1 = \frac{-b - \sqrt{b^2 - 4c}}{2} \qquad \lambda_2 = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

Thus, both $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ solve the ODE (but not the initial data!)

The ODE is linear, so if x_1 and x_2 are solutions, then so is

$$x = A x_1 + B x_2 \qquad \text{for any } A, B \in \mathbb{R}.$$

Then

- $x(0) = A e^{\lambda_1 \cdot 0} + B e^{\lambda_2 \cdot 0} = A + B \overset{!}{=} x_0$
- $x'(0) = \lambda_1 A + \lambda_2 B \overset{!}{=} v_0$

$$\Rightarrow \quad A = \frac{-\lambda_2 x_0 + v_0}{\lambda_1 - \lambda_2}, \qquad B = \frac{\lambda_1 x_0 - v_0}{\lambda_1 - \lambda_2}.$$

This will yield the solution of the initial value problem.

The above breaks down if $\lambda_1 = \lambda_2$!

We have $\lambda_1 = \lambda_2$ iff $b^2 - 4c = 0$, and then $\lambda_1 = \lambda_2 = -\frac{b}{2}$.

$$\boxed{\underline{\text{New ansatz}}: \quad x_1(t) = e^{\lambda t}, \quad x_2(t) = t e^{\lambda t}.}$$

x_1 is a solution as before, and

$$\ddot{x}_2 + b\dot{x}_2 + x_2 = e^{\lambda t}\left(2\lambda + \lambda^2 t + b(1+\lambda t) + ct\right)$$

$$= e^{\lambda t}\left(2\lambda + b + t\left(\lambda^2 + b\lambda + c\right)\right) = 0.$$

As before we set $x(t) = A x_1(t) + B x_2(t)$ and get

$$A = x_0, \quad B = v_0 - x_0 \lambda$$

Proposition

The solution to
$$\begin{cases} \ddot{x} + b\dot{x} + cx = 0 \\ x(0) = x_0, \quad \dot{x}(0) = v_0 \end{cases}$$
is

(a) $x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ if $\lambda_1 \neq \lambda_2$

(b) $x(t) = Ae^{\lambda_1 t} + Bte^{\lambda_1 t}$ if $\lambda_1 = \lambda_2$

(for appropriate $A, B \in \mathbb{C}$). Here, λ_1, λ_2 are the roots of $\lambda^2 + b\lambda + c = 0$.

Interpreting the solution

Recall: If $\lambda \in \mathbb{R}$ then $e^{\lambda t} \begin{cases} \longrightarrow 0 & \text{if } \lambda < 0 \\ \longrightarrow \infty & \text{if } \lambda > 0. \end{cases}$ as $t \to \infty$

- If $\lambda_1, \lambda_2 \in \mathbb{R}$ then $x(t) := Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ is dominated by the term with the largest λ-value:

  if e.g. $\lambda_1 > \lambda_2$ then

$$Ae^{\lambda_1 t} + Be^{\lambda_2 t} = e^{\lambda_1 t}\Big(A + \underbrace{Be^{(\lambda_2 - \lambda_1)t}}_{\approx 0}\Big) \approx Ae^{\lambda_1 t}$$

- If $\lambda_1 = \lambda_2$ and $x(t) = Ae^{\lambda_1 t} + Bte^{\lambda_1 t}$ then

$$x(t) \approx \begin{cases} Ae^{\lambda_1 t} & \text{for small } t \\ Bte^{\lambda_1 t} & \text{for large } t \end{cases}$$

- If $\lambda_1, \lambda_2 \in \mathbb{C}$, say, $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$ for $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ and $i = \sqrt{-1}$ we use Euler's identity:

$$\boxed{e^{ik} = \cos(k) + i\sin(k).}$$

Then

$$x(t) := Ae^{\lambda_1 t} + Be^{\lambda_2 t} = Ae^{(\alpha+\beta i)t} + Be^{(\alpha-\beta i)t}$$

$$= e^{\alpha t}\big(A\cos(\beta t) + Ai\sin(\beta t) + B\cos(-\beta t) + Bi\sin(-\beta t)\big)$$

$$= e^{\alpha t}\left((A+B)\cos(\beta t) + (A-B)\, i \sin(\beta t)\right)$$

- We recall that $A+B = x_0$, and we can show that $A-B = i\gamma$ for some $\gamma \in \mathbb{R}$. Then

$$\boxed{x(t) = e^{\alpha t}\left(x_0 \cos(\beta t) - \gamma \sin(\beta t)\right)}$$

<u>Note</u>:
- The real part α makes the solution <u>grow</u> (if $\alpha > 0$) or <u>shrink</u> (if $\alpha < 0$) as t increases
- The imaginary part β makes the solution <u>oscillate</u> as t increases.

QUESTIONS?

COMMENTS?