UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3500/4500 — Topology

Day of examination: Wednesday, December 17, 2014.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each item (1a, 1b, 2a, etc.) counts 10 points.

Problem 1

Let Y be a topological space. Let X be a non-empty set and let $f : X \to Y$ be a surjective map.

a) Consider the collection τ of all subsets U of X such that $U = f^{-1}(V)$ where V is open in Y. Show that τ is a topology on X. Also show that a subset $A \subset X$ is connected in this topology if and only if $f(A)$ is a connected subset of Y.

b) Show that a subset K of X is compact in τ if and only if $f(K)$ is a compact subset of Y.

c) Show that if $L \subset X$ is closed in τ then $f(L)$ is closed in Y and $L = f^{-1}(f(L)))$.
Let Z be a topological T_1-space (a space where one-point sets are closed). Let $h : X \to Z$ be a map. Show that h is continuous if and only if there exists a continuos map $k : Y \to Z$ such that $k \circ f = h$.

d) Now let $Y = \mathbb{R}$ with standard topology. Let $X = \mathbb{R}^2$, and let f be the map given by $f(x, y) = xy$, where $(x, y) \in \mathbb{R}^2$. Let τ be the topology in $\mathbb{R}^2$ described in a) above. Explain why open sets, respectively closed sets in τ also are open sets, respectively closed sets in the standard topology of $\mathbb{R}^2$. Find an example of a compact set in τ which neither is closed nor bounded in the euclidean metric of $\mathbb{R}^2$. Also find an example of a disconnected set in the standard topology of $\mathbb{R}^2$ which is connected in τ.

(Continued on page 2.)

Problem 2

Let X be a topological space. Suppose that, for every pair of points $x, y \in X$, $x \neq y$, there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$.

a) Show that X is a Hausdorff space.

b) Let K be a non-empty compact subset of X, $K \neq X$. Let $x \in X - K$. Show that there exits a continuous map $f : X \to [0,1]$ such that $f(x) = 1$ and $f(K) \subset [0, \frac{1}{2})$.

Problem 3

A subset A of a topological space X is dense if $\bar{A} = X$.

a) Let U and V be open dense subsets of a space X. Show that $U \cap V$ is dense.

b) Let (X, d) be a metric space, and suppose that X has a countable dense subset. Prove that X is second countable (that X has a countable basis for the metric topology).

Problem 4

In this problem, you can use without proof that if $x \in \mathbb{S}^n$ (here $\mathbb{S}^n$ is the unit n-sphere in $\mathbb{R}^{n+1}$), then there exists a homeomorphism $h : \mathbb{S}^n - \{x\} \to \mathbb{R}^n$

a) Give the definition of a deformation retract A of a topological space X. Give an example of a space X and a proper subset A which is a deformation retract of X. Consider the space $Y = \mathbb{S}^n - \{p, q\}$ where $p = (0, \ldots, 0, 1)$ and $q = (0, \ldots, 0, -1)$. Let $y_0 = (1, 0, \ldots, 0) \in Y$. Find $\pi_1(Y, y_0)$ for all $n > 1$.

b) Let X be a topological space and $f : X \to \mathbb{S}^n$ be a continuous map, $n \geq 1$. Prove that f is nullhomotopic if f is not surjective.

THE END