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Algebra

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Roots of quadratic polynomials

Roots of quadratic polynomials

- The roots, x_1 and x_2, of a quadratic polynomial, $ax^2 + bx + c$, satisfy

$$x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad x_1 x_2 = \frac{c}{a}.$$

Since

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

we get

$$x_1 + x_2 = \frac{-2b}{2a} = \frac{-2b}{2a} \text{ and}$$

$$x_1 x_2 = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Rational roots of polynomial equations

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- If $a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0 = 0$, where $a_i \in \mathbb{Z}$, has a rational root p/q, where $p, q \in \mathbb{Z}$ are relatively prime, then $p \mid a_0$ and $q \mid a_n$. In particular, if $a_n = 1$, the root must be an integer.

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- Proof: We have

$$a_n (p/q)^n + a_{n-1} (p/q)^{n-1} \cdots + a_1 p/q + a_0 = 0 \text{ so}$$
$$a_n p^n + a_{n-1} p^{n-1} q \cdots + a_1 p q^{n-1} + a_0 q^n = 0.$$

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- This gives us

$$a_0 q^n = p(-a_n p^{n-1} - a_{n-1} p^{n-2} q \cdots - a_1 q^{n-1}) \text{ and}$$
$$a_n p^n = q(-a_{n-1} p^{n-1} \cdots - a_1 p q^{n-2} - a_0 q^{n-1}).$$

  This shows that p divides $a_0 q^n$, and since p and q are relatively prime, we must have $p \mid a_0$. It follows similarly that $q \mid a_n$.

$ax^2 + bx + c$ and sliders 1

$ax^2 + bx + c$ and sliders 2

- If you enter $ax^2 + bx + c$ in GeoGebra, it will create sliders. Varying c just moves the graph up and down, but varying a and b makes the graph change in strange ways.

$ax^2 + bx + c$ and sliders 3

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- The extremum of the parabola occurs when $x = -b/(2a)$. You can see this either using calculus, or just observing that if there are two real roots, then this is halfway between them. This is also the symmetry axis.

$ax^2 + bx + c$ and sliders 4

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- Varying b makes the symmetry axis move left and right, in opposite direction to the motion of b, but the graph is also moving up and down.

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- Varying b makes the symmetry axis move left and right, in opposite direction to the motion of b, but the graph is also moving up and down.
- Varying a makes the graph bend up or down from the line $y = bx + c$, which you get when $a = 0$, and increasing $|a|$ makes the graph steeper. However, the graph is both moving up and down and left and right.

$ax^2 + bx + c$ and sliders 6

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- When a goes to $0^{\pm}$, the symmetry axis moves towards $\mp\infty$, and when a goes to ∞, the symmetry axis moves towards the y-axis.

$ax^2 + bx + c$ and sliders 7

$ax^2 + bx + c$ and sliders 8

- The y-value of the extremum is

$$a\left(\frac{-b}{2a}\right)^2 + b\frac{-b}{2a} + c = c + \frac{ab^2 - 2ab^2}{4a^2} = c - b^2/(4a).$$

$ax^2 + bx + c$ and sliders 9

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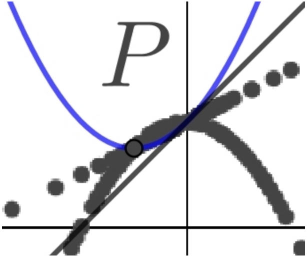
- We can write this as either $c - a(-b/(2a))^2$ or $c - b/2\,(-b/(2a))$, which shows that the extremum lies on the graph of the parabola $y = c - ax^2$ when we vary b and on the graph of the line $y = c + b/2\,x$ when we vary a.



$ax^2 + bx + c$ and sliders 10

$ax^2 + bx + c$ and sliders 11

- Notice how the extremum jumps from ∞ to $-\infty$ when a crosses 0, since we can think of the line as going through a point of infinity that links the two “ends” of the line.

$ax^2 + bx + c$ and sliders 12

- Notice how the extremum jumps from ∞ to $-\infty$ when *a* crosses 0, since we can think of the line as going through a point of infinity that links the two "ends" of the line.
- Notice also that when we vary *a*, the extremum moves on a curve determined by *b* and *c*, while when we vary *b*, the extremum moves on a curve determined by *a* and *c*. This helps explain why we get a parabola in one case and a straight line in the other case.

$ax^2 + bx + c$ and sliders 13

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- Notice also that when we vary a, the extremum moves on a curve determined by b and c, while when we vary b, the extremum moves on a curve determined by a and c. This helps explain why we get a parabola in one case and a straight line in the other case.
- When we vary c, the extremum moves on a curve determined by b and c, but since the x-coordinate is fixed, it is simply a vertical line.

$ax^2 + bx + c$ and sliders 14

$ax^2 + bx + c$ and sliders 15

- Another way to see this is to observe that when $x = -b/(2a)$, then $b = -2ax$, so $f(x) = ax^2 - (2ax)x + c = c - ax^2$, as we saw above.

$ax^2 + bx + c$ and sliders 16

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- We could also write $a = -b/(2x)$, so $f(x) = -bx^2/(2x) + bx + c = b/2\,x + c$, as we also saw above.

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- We could also write $a = -b/(2x)$, so $f(x) = -bx^2/(2x) + bx + c = b/2\, x + c$, as we also saw above.
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Alternative ways of parameterizing parabolas

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- By completing the square, you can also write $ax^2 + bx + c$ as $a(x + p)^2 + q$. If you enter this in GeoGebra and move the sliders, everything is very simple. Varying a simply makes the graph point up or down, and changes the steepness, while varying p and q makes the graph shift horizontally and vertically.