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Analysis

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Cauchy’s false theorem

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- Cauchy's result about series is equivalent to claiming that the limit of a sequence of continuous functions must be continuous. This is true if we require what is known as uniform convergence instead of ordinary pointwise converegence.
- Cauchy was one of the pioners in making analysis rigorous, but even he did not have a clear definition of function, convergence and continuity, and that is why he ended up with a false theorem.

Cauchy's false theorem 2

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- A simple counterexample is $f_n(x) = x^n$ for $x \in [0, 1]$, where

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

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- The reason why the limit is not continuous, is that the rate of convergence becomes slower and slower as we move towards 1. If the rate of convergence is uniform, the theorem can be shown to hold.

A non-analytic function

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- In 1797 Joseph-Louis Lagrange (1736–1813) wrote a book called “Théorie des fonctions analytiques”, where he tried to base analysis on the concept of power series. In 1823 Cauchy constructed an example of a smooth function, i.e., a function that has derivatives of all orders, whose Taylor series does not converge to the function, namely

$$g(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

A non-analytic function 2

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- We will consider the somewhat simpler function

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- Based on Lagrange's terminology, we call a function that is the limit of its Taylor series an analytic function. So f is an example of a non-analytic function.

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- We can prove by induction that for any nonnegative integer n,

$$f^{(n)}(x) = \begin{cases} \frac{p_n(x)}{x^{2n}} f(x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

where $p_n(x)$ is a polynomial of degree $n-1$ for $n > 0$ and $p_0(x) = 1$.

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  where $p_n(x)$ is a polynomial of degree $n-1$ for $n > 0$ and $p_0(x) = 1$.
- We can also prove that for any nonnegative integer m,

$$\lim_{x \to 0} \frac{e^{-1/x}}{x^m} = 0.$$

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- To see this, we use the Taylor series of e^x and observe that for every natural number m (including zero)

$$\frac{1}{x^m} = x\left(\frac{1}{x}\right)^{m+1} \le (m+1)!x\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{x}\right)^n =$$

$$(m+1)!x\exp\left(\frac{1}{x}\right), \quad x > 0,$$

because all the terms in series with $n \neq m+1$ are positive.

$e^{-1/x}$ is non-analytic 2

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- Therefore,

$$0 \leq \lim_{x \to 0^+} \frac{e^{-1/x}}{x^m} \leq \lim_{x \to 0^+} e^{-1/x}(m+1)!xe^{1/x} = (m+1)! \lim_{x \to 0^+} x = 0.$$

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- This implies that for $n \geq 0$

$$f^{n+1}(0) = \lim_{x\to 0^+} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0^+} \frac{p_n(x)}{x^{2n+1}} e^{-1/x} = 0.$$

A continuous function that is nowhere differentiable

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- In 1872 Karl Weierstrass (1815–1897) gave an example of a continuous function that is nowhere differentiable using Fourier series. We will instead give an example given by Teiji Takagi (高木 貞治) (1875–1960) in 1901, which is sometimes called the blancmange curve (named after the dessert).

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- Set $h(x) = \min_{n\in\mathbb{Z}} |x - n|$, $h_n(x) = h(2^n x)/2^n$ and $f(x) = \sum_{n=0}^{\infty} h(2^n x)/2^n$.

A continuous function that is nowhere differentiable 2

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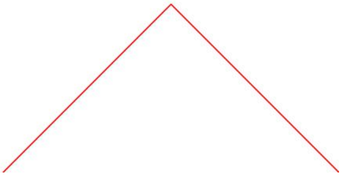
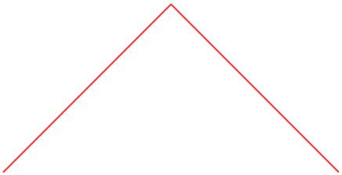


The graph of h_1.

A continuous function that is nowhere differentiable 2



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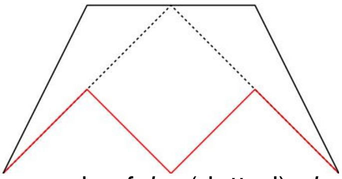

The graph of h_1 (dotted), h_2 and $h_1 + h_2$.

A continuous function that is nowhere differentiable 2

A continuous function that is nowhere differentiable 2

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A continuous function that is nowhere differentiable 2

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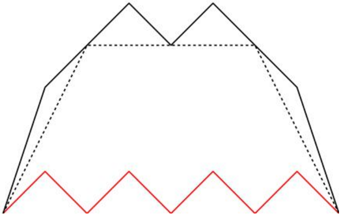
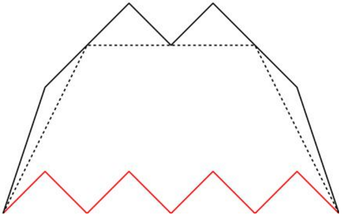
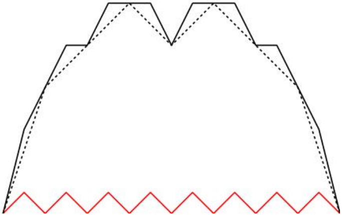


The graph of h_2 (dotted), h_3 and $h_1 + h_2 + h_3$.

A continuous function that is nowhere differentiable 2



The graph of h_2 (dotted), h_3 and $h_1 + h_2 + h_3$.



The graph of h_3 (dotted), h_4 and $h_1 + h_2 + h_3 + h_4$.

A continuous function that is nowhere differentiable 3

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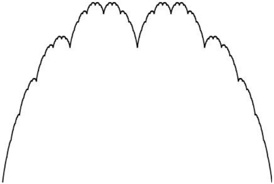
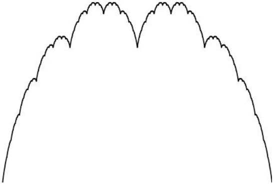
- 

  The graph of h.

A continuous function that is nowhere differentiable 3

- 

  The graph of h.
- It can be shown that the series converges and that the sum is continuous. It is clear that h is not differentiable at points of the form $a/2^b$. Using the sawtooth shape of the curve, it can be shown that the function is not differentiable anywhere!