UiO University of Oslo

Borel — Answers to Sample Questions

Helmer Aslaksen

Dept. of Teacher Education/Dept. of Mathematics
University of Oslo
helmer.aslaksen@gmail.com

Coin Sequence TT-HT

- Start flipping the Borel coin.
Note down the results in the order they appear.
Stop when one of the two sequences appear:
[Heads - Tails] or [Tails - Tails].

Will the sequence [Heads - Tails] appear first?

Coin Sequence TT-HT Solution

- The only way TT can beat HT is if your first two flips are TT, so it follows that.

$$P(\text{YES}) = 3/4.$$

Coin Sequence TT-TH

- Start flipping the Borel coin.
Note down the results in the order they appear.
Stop when one of the two sequences appear:
[Tails - Tails] or [Tails - Heads].

Will the sequence [Tails - Tails] appear first?

Coin Sequence TT-TH Solution 1

- Once you get a T, you will then get T or H the next time, so it is clear that
$$P(\text{YES}) = 1/2.$$
- However, it can be shown that the average waiting time for TH is 4, while the waiting time for TT is 6.
- The point is that the waiting times are actually irrelevant! If you are trying to get TT, and you already have a T, then if you get an H, you have to start again. However, if you are trying to get TH, and get first one T and then another T, you are still just one away from TH.
- If you are trying to get TH, and lose because the first T was followed by another T, you were still just one away from winning. However, if you are trying to get TT, and lose because the first T was followed by an H, you are two away from winning.
- The way I think of this is that with TT, you win half the time, but when you lose, you lose badly.

Coin Sequence TT-TH Solution 2

- Imagine that A and B are taking a number of tests that are scored out of 100 points, with a passing grade of 40. A gets 70 half the time and 30 half the time, while B gets 70 half the time and 10 half the time. A has an average grade of 50, while B has an average grade of 40, but they both pass half the time.
- So while passing rate and average grade in this example are clearly related, they are not the same. Similarly, winning rate and waiting time are not the same.

Coin Sequence TT-TH Solution 3

- Let $E(T)$ be the expected waiting time for the first T. We have $E(T) = E(T|H)P(H) + E(T|T)P(T)$, but if we get H, we must add one to the waiting time, so $E(T|H) = 1 + E(T)$, and we get

$$E(T) = E(T|H)P(H) + E(T|T)P(T)$$
$$= (1 + E(T))/2 + 1/2 = E(T)/2 + 1,$$

  and hence $E(T) = 2$.
- In the same way,

$$E(TH) = E(TH|H)P(H) + E(TH|T)P(T)$$
$$= (1 + E(TH))/2 + (1 + E(H))/2 = E(TH)/2 + 2,$$

  so we get E(TH)=4.

Coin Sequence TT-TH Solution 4

- From

$$
\begin{aligned}
E(TT) &= E(TT|H)P(H) + E(TT|T)P(T) \\
&= (1 + E(TT))/2 + (E(TT|TT) + E(TT|HT))/4 \\
&= E(TT)/2 + 1/2 + (2 + E(TT) + 2)/4 = 3E(TT)/4 + 3/2,
\end{aligned}
$$

we get $E(TT) = 6$.

Coin Stop 2H MoreH

- Keep tossing the Borel coin,
noting the total number of heads and tails you get.
Stop when you reach two heads.

Will you have more heads than tails when you stop?

Coin Stop 2H MoreH Solution

- Consider the first two flips. If you get TT, you will lose. If you get HH, you have won. If you get HT or TH, you will win or lose depending on what you flip next time, so the your chance of winning is 1/2 in both cases. All together, we get

$$P(\text{YES}) = 1/2.$$

Coin Row 8 4Row

- Flip the Borel coin eight times.
  Note the results in the order they appear.

Will you get the same result at least four times in a row?

Coin Row 8 4Row Solution

- Let us first count the number of ways we can get at least four Heads in a row. In this table X denotes either H or T, and you can check that this is a disjoint list of all the possible ways to get at least four Heads in a row.

H^4X^4	16
TH^4X^3	8
XTH^4X^2	8
X^2TH^4X	8
X^3TH^4	8
Total	48

- To find the number of ways we can get the same result four times in a row, we multiply 48 by 2 and subtract 2, since H^4T^4 and T^4H^4 have both been counted twice. Hence

$$P(\text{YES}) = \frac{94}{256} \approx 0.37.$$

Coin Distribution 4 H2

- Flip the Borel coin four times.
  Note the results.

Will you get Heads exactly two times?

Coin Distribution 4 H2 Solution

- Using the binomial distribution, we get

$$P(\text{YES}) = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 0.375.$$

Dice Row D6x8 x2

- Roll a 6-sided die eight times.
Note the results in the order they appear.

Will some number appear at least two times in a row?

Dice Row D6x8 x2 Solution

- Using complimentary probability, we get

$$P(\text{YES}) = 1 - \left(\frac{5}{6}\right)^7 \approx 0.72$$

Dice Stop D6More1 3or4

- Keep rolling a 6-sided die as long as the outcome is above 1.
  Stop when you roll a 1.

  Will a 3 or a 4 have appeared before you stop?

Dice Stop D6More1 3or4 Solution

- The question is simply whether you first roll a 1 or first roll a 3 or 4. It is clear that
$$P(\text{YES}) = \frac{2}{3} \approx 0.67.$$

Dice Sum D6x3D10D20D30 40

- Roll three 6-sided dice, a 10-sided die, a 20-sided die and a 30-sided die.

  Will the sum of the results be at least 40?

Dice Sum D6x3D10D20D30 40 Solution

- The sum of the expected values is

$$3 \cdot 3.5 + 5.5 + 10.5 + 15.5 = 42,$$

and since this is greater than 40, we know that

$$P(\text{YES}) > 0.5.$$

I have not tried to find the exact value.

Dice Increasing D10D6

- Roll a 10-sided die. Then roll a 6-sided die.

  Will the two results be in increasing order?

Dice Increasing D10D6 Solution

	1	2	3	4	5	6	7	8	9	10
1										
2	Y									
3	Y	Y								
4	Y	Y	Y							
5	Y	Y	Y	Y						
6	Y	Y	Y	Y	Y					

Draw a table and observe that there are 15 Yes and $6 + 15 + 24$ No. Hence

$$P(\text{YES}) = \frac{15}{60} = 0.25$$

Dice Increasing D6D10D20

- Roll a 6-sided die.
Then roll a 10-sided die.
Then roll a 20-sided die.

Will the three results be in strictly increasing order?

Dice Increasing D6D10D20 Solution 1

- There are $n = 6 \cdot 10 \cdot 20 = 1200$ possible cases. We need to enumerate the different ways to get strictly increasing results. We will consider four different cases.
  - I: All dice between 1 and 6.
  - II: Two between 1 and 6 and one between 7 and 20.
  - III: One between 1 and 6 and two between 7 and 10.
  - IV: One between 1 and 6, one between 7 and 10 and one between 11 and 20.

Dice Increasing D6D10D20 Solution 2

- Case I: If the D6 is 1, there are $4+3+2+1=10$ possibilities for the remaining two, namely $(2,3),(2,4),...,(2,6),(3,4),...,(3,6),(4,5),(4,6),(5,6)$. If the D6 is 2, there are $3+2+1=6$ possibilities, if the D6 is 3, there are $2+1=3$ possibilities, and if the D6 is 4 there is 1 possibility. So all together 20.
  Case II: The number of upper diagonal entries in a 6×6 matrix is $6\cdot 5/2=15$, which is the number of ways to select the first two, so all together $15\cdot 14=210$.
  Case III: $6\cdot(4\cdot 3/2)=36$.
  Case IV: $6\cdot 4\cdot 10=240$.

$$P(\text{YES})=\frac{20+210+36+240}{1200}=\frac{506}{1200}\approx 0.42.$$

Dice MaxMin D6x3D20 More10

- Roll three 6-sided dice and a 20-sided die.
What is the minimum value you got and what is the maximum?

Will the difference between these two values be greater than 10?

Dice MaxMin D6x3D20 More10 Solution 1

- If the D20 is 11 or less, the difference cannot be greater than 10. So it is clear that $P(\text{YES}) \leq 9/20 = 0.45$.
- To get the exact value, we observe that the total number of ordered cases is $6^3 \cdot 20 = 4320$.
  How many ways can the minimum of the 3 D6 be 1?
  - If the first one is 1, the two others can be anything, so we get $5 \cdot 6 = 36$.
  - If the first one is not 1, but the second one is, we get $5 \cdot 6 = 30$.
  - If the first two are not 1, but the last one is, we get $5 \cdot 5 = 25$, so all together $36 + 30 + 25 = 91$.
  - You can also say that there are $3 \cdot 25 = 75$ ways to get only one 1, $3 \cdot 5 = 15$ ways to get two 1's, and 1 way to get 3 1's, so all together $75 + 15 + 1 = 91$.

Dice MaxMin D6x3D20 More10 Solution 2

- In the same way, you can get a minimum of 2 in $5^2 + 5 \cdot 4 + 4^2 = 61$ ways, 3 in $4^2 + 4 \cdot 3 + 3^2 = 37$, 4 in $3^2 + 3 \cdot 2 + 2^2 = 19$, 5 in $2^2 + 2 \cdot 1 + 1^2 = 7$ and 6 in 1 way. We therefore get that the total number of yes cases is

$$91 \cdot 9 + 61 \cdot 8 + 37 \cdot 7 + 19 \cdot 6 + 7 \cdot 5 + 1 \cdot 4 = 1719.$$

Hence

$$P(\text{YES}) = \frac{1719}{4320} \approx 0.40.$$

Dice Consecutive D6x3 2

- Roll three 6-sided dice.

Will there be two consecutive numbers among the results?
Example: if you roll $[4, 1, 5]$ the answer is Yes.

Dice Consecutive D6x3 2 Solution

- Let us first assume that we roll (ignoring order) (x, x, y), i.e., we roll one number twice. It can be checked that there are then 10 ways to get two consecutive numbers, namely

$$(1,1,2),\ldots,(5,5,6) \quad \text{and} \quad (2,2,1),\ldots,(6,6,5).$$

  We must then multiply by 3 to get 30 such ordered rolls.
- If we instead roll three different numbers, we get $\binom{6}{3} = 20$ possible triples, and we can check that 16 of them will contain two consecutive numbers. We must then multiply by 3! to get 96 such ordered rolls. Hence

$$P(\text{YES}) = \frac{30+96}{6^3} = \frac{126}{216} \approx 0.58.$$

Dice Combination D6x3 2orMore10

- Roll three 6-sided dice.

Will there be a combination of at least two dice that forms exactly sum 10?

Dice Combination D6x3 2orMore10 Solution 1

- We will first see which values of the dice will enable us to form a sum equal to 10. This is the same as looking for all possible partitions of 10 into two or three numbers. We must then count how many possible dice rolls (x, y, z) will give us each of these partitions. That depends on how many different numbers there are among the dice rolls, and the corresponding number of possible permutations of the dice. Let us first see how we can get 10 using three dice.

Partition	Permutations
1+3+6	6
2+2+6	3
2+3+5	6
1+4+5	6
2+2+4	3
3+3+4	3
Total	27

Dice Combination D6x3 2orMore10 Solution 2

- We can get 10 using two dice if we have a partition of the form $(6,4,x)$ or $(5,5,x)$. There will be six of each type. For the first type, four of them will consist of three different numbers, and two of them will consist of two numbers. This will give us $6 \cdot 4 + 3 \cdot 2 = 30$ permutations. For the second type, of the first type and d $5 \cdot 3 + 1 \cdot 1 = 16$. Hence

$$P(\text{YES}) = \frac{27 + 30 + 16}{6^3} = \frac{73}{216} \approx 0.34$$

Pouches Single B6R1 4 R1

- Prepare a pouch with six blue balls and one red ball.
  Take out four balls at random.

  Will the red ball be taken out?

Pouches Single B6R1 4 R1 Solution

- Using complementary probability, we get

$$P(\text{YES}) = 1 - \frac{6}{7}\frac{5}{6}\frac{4}{5}\frac{3}{4} = 1 - \frac{3}{7} \approx 0.57.$$

Pouches Single B2R2 2 R2

- Prepare a pouch with two blue balls and two red balls.
  Take out two balls at random.

  Will both of them be red?

Pouches Single B2R2 2 R2 Solution

- Using the formula for two dependent events, we get

$$P(\text{YES}) = P(R1)P(R2|R1) = \left(\frac{2}{4}\right)\left(\frac{1}{3}\right) = \left(\frac{1}{6}\right) = 0.17.$$

Pouches Single B6R3 2 RM1

- Prepare a pouch with six blue balls and three red balls.
  Draw two balls at random.

  Will at least one of them be red?

Pouches Single B6R3 2 RM1 Solution

- Using complementary probability, we get

$$P(\text{YES}) = 1 - \frac{6}{9}\frac{5}{8} = 1 - \frac{10}{24} \approx 0.58.$$

Pouches Double 5-2,2-5

- Pouch 1 has five blue balls and two red balls, and Pouch 2 has two blue and five red.
  Shuffle the pouches and draw one ball at random.

  Given the colour of the ball, was the Pouch 1 with five blue balls selected?

Pouches Double 5-2,2-5 Solution

- We use Bayes' rule.

$$P(\text{Pouch 1}|\text{Red drawn}) = \frac{P(\text{R}|1)P(1)}{P(\text{R})} =$$

$$\frac{(2/7)(1/2)}{P(\text{R}|1)P(1) + P(\text{R}|2)P(2)} =$$

$$\frac{1/7}{(2/7)(1/2) + (5/7)(1/2)} = \frac{1/7}{1/2} = \frac{2}{7}.$$

- It can be shown that if there is the same total in both bags, then

$$P(\text{Pouch 1}|\text{Red drawn}) = \text{Ratio of red balls in Pouch 1}.$$

Hence

$$P(\text{Pouch 1}|\text{Red drawn}) = \frac{2}{7}$$

$$P(\text{Pouch 1}|\text{Blue drawn}) = \frac{5}{7}.$$

Pouches Triple B1R1,B2,R2 Same

- Prepare three pouches with one blue ball and one red ball, two blue balls, and one red ball and one blue ball, respectively.
  Shuffle the pouches and draw one ball

  Is the ball left in that pouch of the same color?

Pouches Triple B1R1,B2,R2 Same Solution

- Two of the three pouches have two balls of the same color, so

$$P(\text{YES}) = \frac{2}{3} \approx 0.67.$$

Thanks!