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Calculus and Counterexamples

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Limits in high school mathematics

- ▶ To differentiate polynomials, you only need algebra to compute limits. Polynomial differentiation is polynomial division.
- ▶ $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$.
- ▶ Definition of e .
- ▶ The Fundamental Theorem of Calculus

Polynomial differentiation is polynomial division

- ▶ Let $p(x)$ be a polynomial. Then

$$p'(x) = \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h}.$$

We now set $q(h) = p(x+h) - p(x)$, which is a polynomial in h . Since $q(0) = p(x) - p(x) = 0$, we can write $q(h) = hr(h)$, and then

$$p'(x) = \lim_{h \rightarrow 0} \frac{q(h)}{h} = \lim_{h \rightarrow 0} \frac{hr(h)}{h} = \lim_{h \rightarrow 0} r(h) = r(0).$$

Definition of e

- ▶ Does $s_n = \left(1 + \frac{1}{n}\right)^n$ converge?
- ▶ We want to use the fact that a bounded and increasing sequence converges, but it is not clear that s_n is either bounded or increasing.
- ▶ The binomial formula shows that

$$\begin{aligned}
 s_n &= \left(1 + \frac{1}{n}\right)^n \\
 &= 1 + \frac{n}{1!} \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} \\
 &\quad + \dots + \frac{n(n-1)(n-2)\dots 1}{n!} \frac{1}{n^n} \\
 &= 1 + \frac{1}{1!} + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \\
 &\quad + \dots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right).
 \end{aligned}$$

Definition of e 2

- ▶ The product is hard to analyze, since the number of factors increase, while the factors themselves decrease. However, the binomial formula converts s_n to a sum of n terms.
- ▶ Since all the terms in the parenthesis are positive, we have now written s_n as a sum of n positive terms. When we go from s_n to s_{n+1} , terms of the form $(1 - k/n)$ will change to $(1 - k/(n + 1))$, which is larger. So the first n terms increase, and we also add another positive term. It is therefore clear that s_n is increasing.

Definition of e

- ▶ Consider the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ with partial sums

$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

- ▶ Since t_n is obtained from s_n by removing the parenthesis, and all the terms in the parenthesis are less than 1, we see that $s_n \leq t_n$. Since going from t_n to t_{n+1} just adds a positive term, we see that t_n is also increasing.
- ▶ Since

$$n! = 1 \cdot 2 \cdot 3 \cdots n > 1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1},$$

we have

$$s_n \leq t_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.$$

- ▶ It follows that s_n is bounded and increasing, so e exists and $e < 3$.

The Fundamental Theorem of Calculus 1

- ▶ TFC can be stated in two ways. If $f(x)$ is continuous on $[a, b]$, then

$$A(x) = \int_a^x f(t) dt$$

is differentiable and $A'(x) = f(x)$.

- ▶ This can be written as

$$\text{FTC Version 1: } \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

and shows that differentiation and integration are inverse operations.

The Fundamental Theorem of Calculus 2

- ▶ Notice that we should not write

$$A(x) = \int_a^x f(x) dx$$

but use a dummy variable, t , for the integration. Otherwise, we would have to write for instance

$$A(b) = \int_a^b f(b) db,$$

which does not make sense.

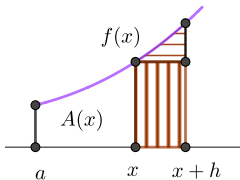
- ▶ Think of the dummy variable as a “hidden”, local variable that is only used for the integration, and x as a global variable that is “seen” by the left hand side.

The Fundamental Theorem of Calculus 3

- ▶ Greek, Islamic, Chinese and Indian mathematicians had throughout the ages solved accumulation problems involving area and volume using ad hoc integration techniques. The reason why this theorem is fundamental, is because it shows that accumulations problems can be solved using anti-differentiation.
- ▶ The FTC changes integration from a bag of tricks to a method that works as long as the function has an anti-derivative.

The Fundamental Theorem of Calculus 4

- To prove the FTC, we observe that $A(x+h) - A(x)$ is the shaded area in the figure, and that if $h \approx 0$, then the area is close to the vertically shaded rectangle. Hence



$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} \approx \lim_{h \rightarrow 0} \frac{f(x)h}{h} = \lim_{h \rightarrow 0} f(x) = f(x).$$

The Fundamental Theorem of Calculus 5

- ▶ We can also state the FTC in a different form by changing the order of the two operations, i.e., we want to show that the integral of the derivative of a function is the function. We therefore consider

$$F(x) = \int_a^x \frac{d}{dt} f(t) dt,$$

and we want to show that

$$\text{FTC Version 2: } \int_a^x \frac{d}{dt} f(t) dt = f(x) - f(a).$$

- ▶ The reason why we get $f(x) - f(a)$ and not just $f(x)$ is because $F(x)$ is defined in terms of a and $F(a) = 0$.

The Fundamental Theorem of Calculus 6

- ▶ The reason why consider the function $F(x)$ and not just the definite integral from a to b is because we want to use the first version of the FTC to prove this version.
- ▶ We know from the first version of the FTC that $F'(x) = f'(x)$, so $F(x) = f(x) + C$ for some constant C , but since $F(a) = f(a) + C = 0$, we get that $F(x) = f(x) - f(a)$.

Product rule 1

- Consider a rectangle of length $f(x)$ and height $g(x)$. The derivative of the product $A(x) = f(x)g(x)$ is the rate of change of the area of the rectangle. We see from the figure that $f(x + \Delta x)g(x + \Delta x) - f(x)g(x)$ splits into three parts.



Product rule 2

- We can also split it into three parts algebraically

$$\begin{aligned}f(x + \Delta x)g(x + \Delta x) - f(x)g(x) &= (f(x + \Delta x) - f(x))g(x) \\ &\quad + f(x)(g(x + \Delta x) - g(x)) \\ &\quad + (f(x + \Delta x) - f(x))(g(x + \Delta x) - g(x)),\end{aligned}$$

and it follows that

$$\begin{aligned}A'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} g(x) \\ &\quad + f(x) \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &\quad + \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} (g(x + \Delta x) - g(x)) \\ &= f'(x)g(x) + f(x)g'(x) + f'(x) \cdot 0 = f'(x)g(x) + f(x)g'(x).\end{aligned}$$

Product rule 3

- ▶ The growth of the area is measured by the growth along the top, which is given by the rate of change of the height times the length plus the growth to the right, which is given by the rate of change of the length times the height.
- ▶ We can ignore the small rectangle in the top right, since both the length and the height goes to zero.

The derivative of $\sin x$ 1

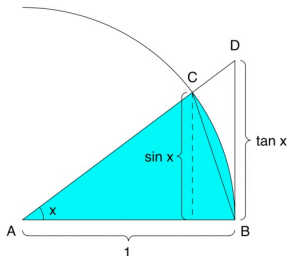
- ▶ To find the derivative of $\sin x$, we consider

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \cos x + \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h}.\end{aligned}$$

- ▶ We want to show that the first limit equals 1 and the second equals 0.

The derivative of $\sin x$ 2

- To find $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ we consider the figure and observe that for small x , the height of the triangle, which equals $\sin x$, is approximately the same as the arc of the circle, which equals x , since we use radians. It follows that the fraction goes to 1.



The derivative of $\sin x$ 3

- ▶ To make this more formal, we observe that the area of the triangle ABC is smaller than the area of the sector ABC , which is smaller than the area of the triangle ABD , which gives us

$$\sin x/2 < x/(2\pi)\pi < \tan x/2,$$

and after multiplying by $2 \cos x / \sin x$ we get

$$\cos x < \cos x \frac{x}{\sin x} < 1.$$

Since $\cos x$ goes to 1, this shows that the fraction also goes to 1.

The derivative of $\sin x$ 4

- ▶ Since

$$\begin{aligned} \frac{\cos x - 1}{x} &= \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \\ &= \frac{-\sin^2 x}{x(\cos x + 1)} = \frac{\sin x}{x} \frac{-\sin x}{\cos x + 1}, \end{aligned}$$

we see that $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 1 \cdot 0 = 0$.

- ▶ This shows that the derivative of $\sin x$ equals $\cos x$. Notice that we cannot use L'Hôpital's rule to compute this limit, since we need this limit to find the derivative of \sin .

Linear approximation 1

- ▶ A basic idea of calculus is to approximate a function $f(x)$ with its tangent line at a point $x = a$,

$$y = f(a) + f'(a)(x - a).$$

- ▶ As an example, let us try to estimate $\sqrt{10}$ using linear approximation around $x = 9$. If $f(x) = \sqrt{x}$, then $f'(x) = 1/(2\sqrt{x})$, so

$$\sqrt{10} = f(10) \approx f(9) + f'(9)(10 - 9) = 3 + 1/6 \approx 3.167.$$

We have $\sqrt{10} \approx 3.162$, so it is a very good approximation.

L'Hôpital's Rule

- ▶ Let f and g be continuous on an interval containing a , and assume f and g are differentiable on this interval with the possible exception of the point a . If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L,$$

for $L \in \mathbb{R} \cup \infty$.

- ▶ The idea behind the proof is to replace the functions with their tangent lines. Since $f(a) = g(a) = 0$, we have

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} = \frac{f'(a)}{g'(a)}$$

- ▶ $f: U \rightarrow \mathbb{R}$ is continuous at $a \in U$ if $\lim_{x \rightarrow a} f(x) = f(a)$ and continuous on U if it is continuous at all points in U .
- ▶ Some people say that f is continuous if and only if we can draw the graph of f without lifting the pen. However, $f(x) = 1/x$ is continuous on $U = \mathbb{R} - \{0\}$.

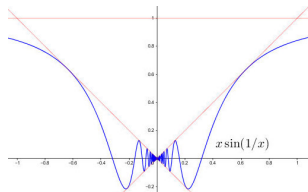
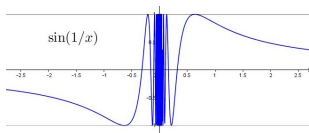
Continuity and differentiability

- ▶ Can you give an example of a function that is continuous but not differentiable?
- ▶ Can you give an example of such a function that has a tangent?
- ▶ Differentiability does not mean having a tangent line, but having a tangent line with a finite slope.
- ▶ A function with a vertical tangent is not differentiable. How do you construct such a function?
- ▶ Invert a function with a horizontal tangent, so take for instance $f(x) = x^{1/3}$.

Source of counterexamples 1



$$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



- ▶ f_0 is not continuous, since $\lim_{x \rightarrow 0} f_0(x)$ does not exist. However, $\lim_{x \rightarrow 0} f_1(x) = 0$, so f_1 is continuous.

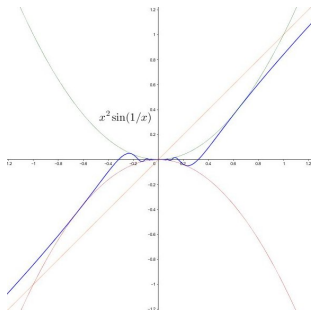
Can you draw the graph of a continuous function?

- ▶ Do you tell your students that a function is continuous if you can draw the graph without lifting the pen?
- ▶ Is $1/x$ continuous?
- ▶ Is it true if you assume that the domain is connected?
- ▶ How long is the graph of $x \sin(1/x)$?
- ▶ Set $x_i = 1 / ((i + 1/2)\pi)$. Then $f_1(x_i) = (-1)^i / ((i + 1/2)\pi)$.
- ▶ Join the points $(x_i, f_1(x_i))$ with lines, starting with $i = 1$ and ending at $i = n$. The distance between $(x_i, f_1(x_i))$ and $(x_{i+1}, f_1(x_{i+1}))$ is bigger than the vertical distance between them, which is bigger than $2|f_1(x_{i+1})| = 2 / ((i + 1 + 1/2)\pi) > 2 / ((i + 2)\pi)$.
- ▶ But since the harmonic series $\sum 1/i$ diverges, which can be shown without calculus, the length of the lines from $(x_1, f_1(x_1))$ to $(x_n, f_1(x_n))$ will go to infinity, and it follows that the arc length of f_1 is infinite.

More counterexamples 1



$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- ▶ Notice that $\lim_{x \rightarrow 0} f'(x)$ does not exist, so f' is not continuous!

More counterexamples 2

- ▶ There is a theorem due to Darboux, which says that a derivative has the intermediate value property.
- ▶ This implies that a derivative cannot have a jump discontinuity, so that if f is not continuously differentiable, then the discontinuity must be an essential discontinuity, i.e., at least one one-sided limit does not exist.

Monotonicity 1

- ▶ Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

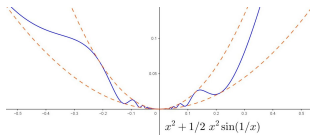
- ▶ $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b) .
- ▶ $f' \geq 0$ on $(a, b) \implies f$ is increasing on (a, b) .
- ▶ $f' \geq 0$ on $(a, b) \iff f$ is increasing on (a, b) .
- ▶ $f(x) = x^3$ shows that $f' \geq 0$ on $(a, b) \not\iff f$ is strictly increasing on (a, b) .

Monotonicity 2

- ▶ If f' is positive on (a, b) , then f is increasing on (a, b) . But what if we only know that $f'(c) > 0$? Can we say that f is increasing on an interval around c ?
- ▶ $f(x) = x + 2x^2 \sin(1/x)$,
 $f'(x) = 1 + 4x \sin(1/x) - 2 \cos(1/x)$ is both positive and negative in every neighborhood of 0.

Extreme point

- ▶ If c is an extreme point and $f'(c)$ exists, then $f'(c) = 0$.
- ▶ First Derivative Test: If f' exists around c , and f' changes sign at c , then c is an extreme point.
- ▶ Second Derivative Test: If $f'(c) = 0$ and $f''(c)$ is positive (negative), then c is a minimum (maximum).
- ▶ We will now see that the converse to the First Derivative is not always true.
- ▶ $f(x) = x^2(1 + 1/2 \sin(1/x))$,
 $f'(x) = 2x + x \sin(1/x) - 1/2 \cos(1/x)$.



- ▶ The reason for the $1/2$ factor, is that $x^2 + x^2 \sin(1/x)$ has infinitely many zeros, which makes 0 a non-isolated extremum.

Point of inflection 1

- ▶ We say that c is a point of inflection if f has a tangent line at c and f'' changes sign at c . (Some people only require that f should be continuous at c .)
- ▶ $f(x) = x^3$ has $f'(0) = 0$, but 0 is not an extremum, but a point of inflection.
- ▶ $f(x) = x^3 + x$ shows that f' does not have to be 0 at a point of inflection.

Point of inflection 2

- ▶ $f(x) = x^{1/3}$ has a point of inflection at 0, has a tangent line at 0, but $f'(0)$ and $f''(0)$ do not exist. (Vertical tangent line. Just bend a bit, and you get a point of inflection.)



$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0, \end{cases}$$

has a point of inflection at 0, and $f'(0)$ exists, but $f''(0)$ does not exist. (First derivatives match, so we get a tangent line, but second derivatives do not match.)

Point of inflection 3

1. If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.
2. If c is a point of inflection, then c is an isolated extremum of f' .
3. If c is a point of inflection, then the curve lies on different sides of the tangent line at c .

Point of inflection 4

- ▶ Proof of 3: We use MVT to get x_1 between c and x with

$$\frac{f(x) - f(c)}{x - c} = f'(x_1),$$

or

$$f(x) = f(c) + f'(x_1)(x - c).$$

- ▶ We now use MVT again to get x_2 between c and x_1 with

$$\frac{f'(x_1) - f'(c)}{x_1 - c} = f''(x_2),$$

or

$$f'(x_1) = f'(c) + f''(x_2)(x_1 - c).$$

- ▶ Combining this, we get

$$\begin{aligned} f(x) &= f(c) + f'(x_1)(x - c) \\ &= f(c) + f'(c)(x - c) + f''(x_2)(x - c)(x_1 - c). \end{aligned}$$

Point of inflection 5

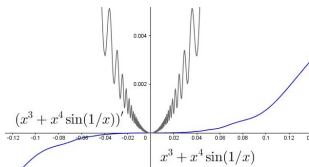
- ▶ The tangent line to $f(x)$ at c is $t(x) = f(c) + f'(c)(x - c)$, so the distance between f and the tangent is $f''(x_2)(x - c)(x_1 - c)$.
- ▶ Since $(x_1 - c)$ and $(x_2 - c)$ have the same sign, their product is positive. But $f''(x)$ changes sign at c , so $f(x)$ will lie on different sides of the tangent at c .

Point of inflection 6

- ▶ Converse to 1 is false: $f(x) = x^4$ has $f''(0) = 0$, but $f''(x) \geq 0$.
- ▶ Converse to 2 is false: $f(x) = x^3 + x^4 \sin(1/x)$ has

$$\begin{aligned} f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x)) \geq 0 \end{aligned}$$

in a neighborhood of 0, so 0 is an isolated minimum of $f'(x)$. We have $f''(0) = 0$, but $f''(x) = 6x - \sin(1/x) - 6x \cos(1/x) + 12x^2 \sin(1/x)$ does not change sign.



Point of inflection 7

- ▶ We need to “integrate” the example $2x^2 + x^2 \sin(1/x)$. Since the derivative of $1/x$ is $-1/x^2$, we try

$$f(x) = x^3 + x^4 \sin(1/x),$$

$$\begin{aligned} f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x)). \end{aligned}$$

- ▶ The first two terms give us the shape we want, and the last terms is so small that we can ignore it.

Point of inflection 8

- Converse to 3 is false:

$f(x) = 2x^3 + x^3 \sin(1/x) = x^3(2 + \sin(1/x))$ lies below the tangent ($y = 0$) on one side and above the tangent on another, but

$f''(x) = 12x + 6x \sin(1/x) - 4 \cos(1/x) - (1/x) \sin(1/x)$ does not change sign, since when x is small, the last term will be oscillate wildly.

- The cubic terms gives the desired shape of the curve, and since the derivative of $1/x$ is $-1/x^2$, we will get a term of the form $(1/x) \sin(1/x)$ in $f''(x)$, which will make it oscillate wildly.

