UiO : University of Oslo

Calculus and Counterexamples

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- The Fundamental Theorem of Calculus

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- Let $p(x)$ be a polynomial. Then

$$p'(x) = \lim_{h\to 0} \frac{p(x+h)-p(x)}{h}.$$

We now set $q(h) = p(x+h) - p(x)$, which is a polynomial in h. Since $q(0) = p(x) - p(x) = 0$, we can write $q(h) = hr(h)$, and then

$$p'(x) = \lim_{h\to 0} \frac{q(h)}{h} = \lim_{h\to 0} \frac{hr(h)}{h} = \lim_{h\to 0} r(h) = r(0).$$

Definition of e 1

Definition of e 2

- Does $s_n = \left(1 + \frac{1}{n}\right)^n$ converge?

Definition of e 3

- Does $s_n = \left(1 + \frac{1}{n}\right)^n$ converge?
- We want to use the fact that a bounded and increasing sequence converges, but it is not clear that s_n is either bounded or increasing.

Definition of e 4

- Does $s_n = \left(1+\frac{1}{n}\right)^n$ converge?
- We want to use the fact that a bounded and increasing sequence converges, but it is not clear that s_n is either bounded or increasing.
- The binomial formula shows that

$$
\begin{aligned}
s_n &= \left(1+\frac{1}{n}\right)^n \\
&= 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} \\
&\quad + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}\frac{1}{n^n} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\
&\quad + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).
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$$

Definition of e 5

Definition of e 6

- The product is hard to analyze, since the number of factors increase, while the factors themselves decrease. However, the binomial formula converts s_n to a sum of n terms.

Definition of e 7

- The product is hard to analyze, since the number of factors increase, while the factors themselves decrease. However, the binomial formula converts s_n to a sum of n terms.
- Since all the terms in the parenthesis are positive, we have now written s_n as a sum of n positive terms. When we go from s_n to s_{n+1}, terms of the form $(1 - k/n)$ will change to $(1 - k/(n+1))$, which is larger. So the first n terms increase, and we also add another positive term. It is therefore clear that s_n is increasing.

Definition of e 8

Definition of e 9

- Consider the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ with partial sums

$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

Definition of e 10

- Consider the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ with partial sums

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- Since t_n is obtained from s_n by removing the parenthesis, and all the terms in the parenthesis are less than 1, we see that $s_n \leq t_n$. Since going from t_n to t_{n+1} just adds a positive term, we see that t_n is also increasing.

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- Since

$$n! = 1 \cdot 2 \cdot 3 \ldots n > 1 \cdot 2 \cdot 2 \ldots 2 = 2^{n-1},$$

we have

$$s_n \leq t_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.$$

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- It follows that s_n is bounded and increasing, so e exists and $e < 3$.

The Fundamental Theorem of Calculus 1

The Fundamental Theorem of Calculus 2

- TFC can be stated in two ways. If $f(x)$ is continuous on $[a, b]$, then

$$A(x) = \int_a^x f(t)\,dt$$

is differentiable and $A'(x) = f(x)$.

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- This can be written as

$$\text{FTC Version 1: } \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

and shows that differentiation and integration are inverse operations.

The Fundamental Theorem of Calculus 4

The Fundamental Theorem of Calculus 5

- Notice that we should not write

$$A(x) = \int_a^x f(x)\,dx$$

but use a dummy variable, t, for the integration. Otherwise, we would have to write for instance

$$A(b) = \int_a^b f(b)\,db,$$

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The Fundamental Theorem of Calculus 6

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- Think of the dummy variable as a “hidden", local variable that is only used for the integration, and x as a global variable that is “seen" by the left hand side.

The Fundamental Theorem of Calculus 7

The Fundamental Theorem of Calculus 8

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The Fundamental Theorem of Calculus 9

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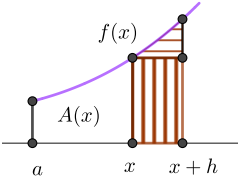
- The FTC changes integration from a bag of tricks to a method that works as long as the function has an anti-derivative.

The Fundamental Theorem of Calculus 10

The Fundamental Theorem of Calculus 11

- To prove the FTC, we observe that $A(x+h) - A(x)$ is the shaded area in the figure, and that if $h \approx 0$, then the area is close to the vertically shaded rectangle. Hence



$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \approx \lim_{h\to 0} \frac{f(x)h}{h} = \lim_{h\to 0} f(x) = f(x).$$

The Fundamental Theorem of Calculus 12

The Fundamental Theorem of Calculus 13

- We can also state the FTC in a different form by changing the order of the two operations, i.e., we want to show that the integral of the derivative of a function is the function. We therefore consider

$$F(x) = \int_a^x \frac{d}{dt} f(t)\, dt,$$

and we want to show that

$$\text{FTC Version 2: } \int_a^x \frac{d}{dt} f(t)\, dt = f(x) - f(a).$$

The Fundamental Theorem of Calculus 14

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- The reason why we get $f(x) - f(a)$ and not just $f(x)$ is because $F(x)$ is defined in terms of a and $F(a) = 0$.

The Fundamental Theorem of Calculus 15

The Fundamental Theorem of Calculus 16

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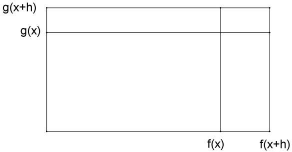
The Fundamental Theorem of Calculus 17

- The reason why consider the function $F(x)$ and not just the definite integral from a to b is because we want to use the first version of the FTC to prove this version.
- We know from the first version of the FTC that $F'(x) = f'(x)$, so $F(x) = f(x) + C$ for some constant C, but since $F(a) = f(a) + C = 0$, we get that $F(x) = f(x) - f(a)$.

Product rule 1

Product rule 2

- Consider a rectangle of length $f(x)$ and height $g(x)$. The derivative of the product $A(x) = f(x)g(x)$ is the rate of change of the area of the rectangle. We see from the figure that $f(x+\Delta x)g(x+\Delta x) - f(x)g(x)$ splits into three parts.



Product rule 3

Product rule 4

- We can also split it into three parts algebraically

$$\begin{aligned} f(x+\Delta x)g(x+\Delta x) - f(x)g(x) = {} & (f(x+\Delta x) - f(x))g(x) \\ & + f(x)(g(x+\Delta x) - g(x)) \\ & + (f(x+\Delta x) - f(x))(g(x+\Delta x) - g(x)), \end{aligned}$$

and it follows that

$$\begin{aligned} A'(x) = {} & \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} g(x) \\ & + f(x) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ & + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} (g(x+\Delta x) - g(x)) \\ = {} & f'(x)g(x) + f(x)g'(x) + f'(x) \cdot 0 = f'(x)g(x) + f(x)g'(x). \end{aligned}$$

Product rule 5

Product rule 6

- The growth of the area is measured by the growth along the top, which is given by the rate of change of the height times the length plus the growth to the right, which is given by the rate of change of the length times the height.

Product rule 7

- The growth of the area is measured by the growth along the top, which is given by the rate of change of the height times the length plus the growth to the right, which is given by the rate of change of the length times the height.
- We can ignore the small rectangle in the top right, since both the length and the height goes to zero.

The derivative of $\sin x$ 1

The derivative of $\sin x$ 2

- To find the derivative of $\sin x$, we consider

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin h}{h}\cos x+\sin x\lim_{h\to 0}\frac{\cos h-1}{h}.$$

The derivative of $\sin x$ 3

- To find the derivative of $\sin x$, we consider

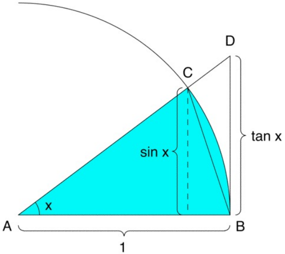
$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin h}{h}\cos x+\sin x\lim_{h\to 0}\frac{\cos h-1}{h}.\end{aligned}$$

- We want to show that the first limit equals 1 and the second equals 0.

The derivative of $\sin x$ 4

The derivative of $\sin x$ 5

- To find $\lim_{x \to 0} \frac{\sin x}{x}$ we consider the figure and observe that for small x, the height of the triangle, which equals $\sin x$, is approximately the same as the arc of the circle, which equals x, since we use radians. It follows that the fraction goes to 1.



The derivative of $\sin x$ 6

The derivative of $\sin x$ 7

- To make this more formal, we observe that the area of the triangle *ABC* is smaller than the area of the sector *ABC*, which is smaller than the area of the triangle *ABD*, which gives us

$$\sin x/2 < x/(2\pi)\pi < \tan x/2,$$

and after multiplying by $2\cos x/\sin x$ we get

$$\cos x < \cos x\frac{x}{\sin x} < 1.$$

Since $\cos x$ goes to 1, this shows that the fraction also goes to 1.

The derivative of $\sin x$ 8

The derivative of $\sin x$ 9

- Since

$$\frac{\cos x - 1}{x} = \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} = \frac{\cos^2 x - 1}{x(\cos x + 1)} =$$

$$\frac{-\sin^2 x}{x(\cos x + 1)} = \frac{\sin x}{x} \frac{-\sin x}{\cos x + 1},$$

we see that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 1 \cdot 0 = 0$.

The derivative of $\sin x$ 10

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  we see that $\lim_{x \to 0} \frac{\cos x - 1}{x} = 1 \cdot 0 = 0$.
- This shows that the derivative of $\sin x$ equals $\cos x$. Notice that we cannot use L'Hôpital's rule to compute this limit, since we need this limit to find the derivative of $\sin$.

Linear approximation 1

Linear approximation 2

- A basic idea of calculus is to approximate a function $f(x)$ with its tangent line at a point $x = a$,

$$y = f(a) + f'(a)(x - a).$$

Linear approximation 3

- A basic idea of calculus is to approximate a function $f(x)$ with its tangent line at a point $x = a$,

$$y = f(a) + f'(a)(x - a).$$

- As an example, let us try to estimate $\sqrt{10}$ using linear approximation around $x = 9$. If $f(x) = \sqrt{x}$, then $f'(x) = 1/(2\sqrt{x})$, so

$$\sqrt{10} = f(10) \approx f(9) + f'(9)(10 - 9) = 3 + 1/6 \approx 3.167.$$

  We have $\sqrt{10} \approx 3.162$, so it is a very good approximation.

L'Hôpital's Rule

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- Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,$$

for $L \in \mathbb{R} \cup \infty$.

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- The idea behind the proof is to replace the functions with their tangent lines. Since $f(a) = g(a) = 0$, we have

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a))} = \frac{f'(a)}{g'(a))}$$

Continuity

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- $f\colon U \to \mathbb{R}$ is continuous at $a \in U$ if $\lim_{x\to a} f(x) = f(a)$ and continuous on U if it is continuous at all points in U.

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- $f\colon U \to \mathbb{R}$ is continuous at $a \in U$ if $\lim_{x\to a} f(x) = f(a)$ and continuous on U if it is continuous at all points in U.
- Some people say that f is continuous if and only if we can draw the graph of f without lifting the pen. However, $f(x) = 1/x$ is continuous on $U = \mathbb{R} - \{0\}$.

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Continuity and differentiability

- Can you give an example of a function that is continuous but not differentiable?
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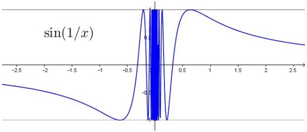
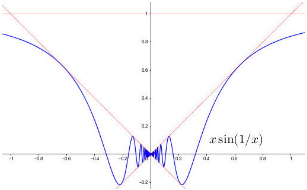
- Differentiability does not mean having a tangent line, but having a tangent line with a finite slope.
- A function with a vertical tangent is not differentiable. How do you construct such a function?
- Invert a function with a horizontal tangent, so take for instance $f(x) = x^{1/3}$.

Source of counterexamples 1

Source of counterexamples 2

- $$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$





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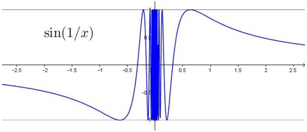

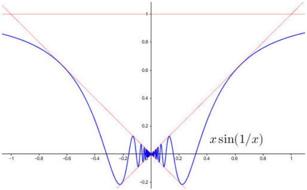

- f_0 is not continuous, since $\lim_{x\to 0} f_0(x)$ does not exist. However, $\lim_{x\to 0} f_1(x) = 0$, so f_1 is continuous.

Can you draw the graph of a continuous function?

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- How long is the graph of $x\sin(1/x)$?
- Set $x_i = 1/\left((i+1/2)\pi\right)$. Then $f_1(x_i) = (-1)^i/((i+1/2)\pi)$.

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- How long is the graph of $x \sin(1/x)$?
- Set $x_i = 1/((i+1/2)\pi)$. Then $f_1(x_i) = (-1)^i/((i+1/2)\pi)$.
- Join the points $(x_i, f_1(x_i))$ with lines, starting with $i = 1$ and ending at $i = n$. The distance between $(x_i, f_1(x_i))$ and $(x_{i+1}, f_1(x_{i+1}))$ is bigger than the vertical distance between them, which is bigger than $2|f_1(x_{i+1})| = 2/((i+1+1/2)\pi) > 2/((i+2)\pi)$.

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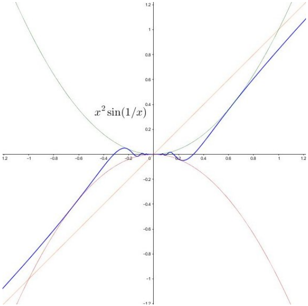
- But since the harmonic series $\sum 1/i$ diverges, which can be shown without calculus, the length of the lines from $(x_1, f_1(x_1))$ to $(x_n, f_1(x_n)$ will go to infinity, and it follows that the arc length of f_1 is infinite.

More counterexamples 1

More counterexamples 2

- $$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



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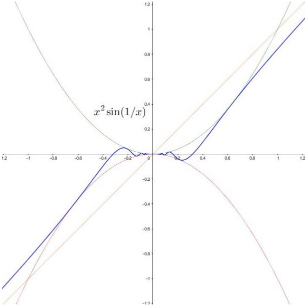

- $$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

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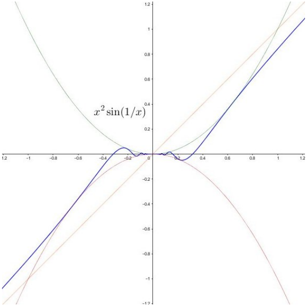

- $$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
- Notice that $\lim_{x \to 0} f'(x)$ does not exist, so f' is not continuous!

More counterexamples 5

More counterexamples 6

- There is a theorem due to Darboux, which says that a derivative has the intermediate value property.

More counterexamples 7

- There is a theorem due to Darboux, which says that a derivative has the intermediate value property.
- This implies that a derivative cannot have a jump discontinuity, so that if f is not continuously differentiable, then the discontinuity must be an essential discontinuity, i.e., at least one one-sided limit does not exist.

Monotonicity 1

Monotonicity 2

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Monotonicity 3

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$
- $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b).

Monotonicity 4

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$
- $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b).
- $f' \geq 0$ on $(a, b) \implies f$ is increasing on (a, b).

Monotonicity 5

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
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Monotonicity 6

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
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- $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b).
- $f' \geq 0$ on $(a, b) \implies f$ is increasing on (a, b).
- $f' \geq 0$ on $(a, b) \impliedby f$ is increasing on (a, b).
- $f(x) = x^3$ shows that $f' \geq 0$ on $(a, b) \not\Longleftarrow f$ is strictly increasing on (a, b).

Monotonicity 7

Monotonicity 8

- If f' is positive on (a, b), then f is increasing on (a, b). But what if we only know that $f'(c) > 0$? Can we say that f is increasing on an interval around c?

Monotonicity 9

- If f' is positive on (a, b), then f is increasing on (a, b). But what if we only know that $f'(c) > 0$? Can we say that f is increasing on an interval around c?
- $f(x) = x + 2x^2 \sin(1/x)$,
  $f'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x)$ is both positive and negative in every neighborhood of 0.

Extreme point

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- If c is an extreme point and $f'(c)$ exists, then $f'(c) = 0$.

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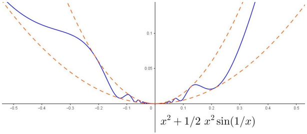
- $f(x) = x^2(1 + 1/2\sin(1/x))$,
  $f'(x) = 2x + x\sin(1/x) - 1/2\cos(1/x)$.



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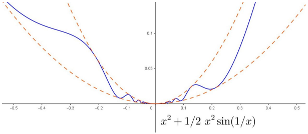

- The reason for the $1/2$ factor, is that $x^2 + x^2 \sin(1/x))$ has infinitely many zeros, which makes 0 a non-isolated extremum.

Point of inflection 1

Point of inflection 2

- We say that c is a point of inflection if f has a tangent line at c and f'' changes sign at c. (Some people only require that f should be continuous at c.)

Point of inflection 3

- We say that c is a point of inflection if f has a tangent line at c and f'' changes sign at c. (Some people only require that f should be continuous at c.)
- $f(x) = x^3$ has $f'(0) = 0$, but 0 is not an extremum, but a point of inflection.

Point of inflection 4

- We say that c is a point of inflection if f has a tangent line at c and f'' changes sign at c. (Some people only require that f should be continuous at c.)
- $f(x) = x^3$ has $f'(0) = 0$, but 0 is not an extremum, but a point of inflection.
- $f(x) = x^3 + x$ shows that f' does not have to be 0 at a point of inflection.

Point of inflection 5

Point of inflection 6

- $f(x) = x^{1/3}$ has a point of inflection at 0, has a tangent line at 0, but $f'(0)$ and $f''(0)$ do not exist. (Vertical tangent line. Just bend a bit, and you get a point of inflection.)

Point of inflection 7

- $f(x) = x^{1/3}$ has a point of inflection at 0, has a tangent line at 0, but $f'(0)$ and $f''(0)$ do not exist. (Vertical tangent line. Just bend a bit, and you get a point of inflection.)
-
$$f(x) = \begin{cases} x^2 \text{ if } x \geq 0, \\ -x^2 \text{ if } x < 0, \end{cases}$$

  has a point of inflection at 0, and $f'(0)$ exists, but $f''(0)$ does not exist. (First derivatives match, so we get a tangent line, but second derivatives do not match.)

Point of inflection 8

Point of inflection 9

1. If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.

Point of inflection 10

1. If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.
2. If c is a point of inflection, then c is an isolated extremum of f'.

Point of inflection 11

1. If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.
2. If c is a point of inflection, then c is an isolated extremum of f'.
3. If c is a point of inflection, then the curve lies on different sides of the tangent line at c.

Point of inflection 12

Point of inflection 13

- Proof of 3: We use MVT go get x_1 between c and x with

$$\frac{f(x) - f(c)}{x - c} = f'(x_1),$$

or

$$f(x) = f(c) + f'(x_1)(x - c).$$

Point of inflection 14

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- We now use MVT again to get x_2 between c and x_1 with

$$\frac{f'(x_1) - f'(c)}{x_1 - c} = f''(x_2),$$

or

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Point of inflection 15

- Proof of 3: We use MVT go get x_1 between c and x with

$$\frac{f(x)-f(c)}{x-c}=f'(x_1),$$

or

$$f(x)=f(c)+f'(x_1)(x-c).$$

- We now use MVT again to get x_2 between c and x_1 with

$$\frac{f'(x_1)-f'(c)}{x_1-c}=f''(x_2),$$

or

$$f'(x_1)=f'(c)+f''(x_2)(x_1-c).$$

- Combining this, we get

$$\begin{aligned}f(x)&=f(c)+f'(x_1)(x-c)\\&=f(c)+f'(c)(x-c)+f''(x_2)(x-c)(x_1-c).\end{aligned}$$

Point of inflection 16

Point of inflection 17

- The tangent line to $f(x)$ at c is $t(x) = f(c) + f'(c)(x - c)$, so the distance between f and the tangent is $f'(x_2)(x - c)(x_1 - c)$.

Point of inflection 18

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- Since $(x_1 - c)$ and $(x_1 - c)$ have the same sign, their product is positive. But $f''(x)$ changes sign at c, so $f(x)$ will lie on different sides of the tangent at c.

Point of inflection 19

Point of inflection 20

- Converse to 1 is false: $f(x) = x^4$ has $f''(0) = 0$, but $f''(x) \geq 0$.

Point of inflection 21

- Converse to 1 is false: $f(x) = x^4$ has $f''(0) = 0$, but $f''(x) \geq 0$.
- Converse to 2 is false: $f(x) = x^3 + x^4 \sin(1/x)$ has

$$\begin{aligned} f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x)) \geq 0 \end{aligned}$$

  in a neighborhood of 0, so 0 is an isolated minimum of $f'(x)$. We have $f''(0) = 0$, but $f''(x) = 6x - \sin(1/x) - 6x \cos(1/x) + 12x^2 \sin(1/x)$ does not change sign.

Point of inflection 22

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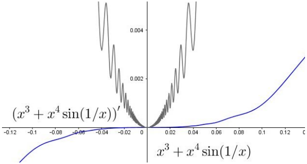
- 

Point of inflection 23

Point of inflection 24

- We need to "integrate" the example $2x^2 + x^2 \sin(1/x)$. Since the derivative of $1/x$ is $-1/x^2$, we try

$$\begin{aligned} f(x) &= x^3 + x^4 \sin(1/x), \\ f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x)). \end{aligned}$$

Point of inflection 25

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- The first two terms give us the shape we want, and the last terms is so small that we can ignore it.

Point of inflection 26

Point of inflection 27

- Converse to 3 is false:
  $f(x) = 2x^3 + x^3 \sin(1/x) = x^3(2 + \sin(1/x))$ lies below the tangent ($y = 0$) on one side and above the tangent on another, but
  $f''(x) = 12x + 6x \sin(1/x) - 4 \cos(1/x) - (1/x) \sin(1/x)$
  does not change sign, since when x is small, the last term will be oscillate wildly.

Point of inflection 28

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- The cubic terms gives the desired shape of the curve, and since the derivative of $1/x$ is $-1/x^2$, we will get a term of the form $(1/x)\sin(1/x)$ in $f''(x)$, which will make it oscillate wildly.

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