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Decimal Expansion of Rational Numbers

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What is this about?

- Q1: Why is $0.999\ldots = 0.\overline{9} = 1$?
- Q2: Consider $1/2 = 0.5$, $1/3 = 0.\overline{3}$ and $1/6 = 0.1\overline{6}$.
- The first has a finite, terminating decimal expansion. The second has a repeating block that starts right after the decimal point, while the third has a finite part before the repeating block.
- We will see that any rational number will have a decimal expansion of one of these three types, and that the type only depends on the numerator, so we will only consider unit fractions of the form $1/n$. Can you tell which type $1/n$ will be?
- Q3: How does the length of the repeating block depend on n?
- Notice how $1/7 = 0.\overline{142857}$ has $7 - 1 = 6$ digits in the repeating block, while $1/3 = 0.\overline{3}$ has 1, and not $3 - 1 = 2$ digits in the repeating block.
- Q4: Is there a pattern in $1/7 = 0.\overline{142857}$?

Why is this interesting?

- This is a topic I discuss in my course on “School Mathematics from an Advanced Viewpoint” at the University of Oslo.
- The results are classical, but not well-known.
- The questions are easy to ask, but difficult to answer.
- Students might ask you about this, and you should know how to answer it.
- You may use this as an exploratory exercise.
- It gives an interesting application of number theory.
- It is a good motivation for why students need to understand long division.

Why is $0.999\ldots = 1$? 1

- We can multiply $1/3 = 0.\overline{3}$ by 3 and get

$$3 \cdot 1/3 = 1 \quad \text{and} \quad 3 \cdot 0.\overline{3} = 0.\overline{9}.$$

- We can also write

$$\begin{aligned} x &= 0.\overline{9}, \\ 10x &= 9.\overline{9}, \\ 9x &= 10x - x = 9.\overline{9} - 0.\overline{9} = 9, \\ x &= 1. \end{aligned}$$

Why is $0.999\ldots = 1$? 2

- Since $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ for $|x| < 1$, we have

$$
\begin{aligned}
0.\overline{9} &= 9(0.1 + 0.01 + 0.001 + \cdots) \\
&= 9(0.1 + 0.1^2 + 0.1^3 + \cdots) \\
&= 9\sum_{k=1}^{\infty} 0.1^k = 9 \cdot 0.1 \sum_{k=0}^{\infty} 0.1^k \\
&= 0.9\frac{1}{1-0.1} = \frac{0.9}{0.9} = 1.
\end{aligned}
$$

Why is $0.999\ldots = 1$? 3

- It is important to point out that $x = x_0.d_1d_2\ldots$ is the limit of $x_n = x_0.d_1d_2\ldots d_n$. It is not a finite decimal number.
- Most people will agree that the sequence $0.9, 0.99, \ldots$ approaches 1, but that is exactly what $0.\overline{9} = 1$ means.

Why is $0.999\ldots = 1$? 4

- Finally, we can argue that they have to be equal, since if they were not equal, we could find some number between them. However, there is no way to put any number between them.
- Similarly, we can show that

$$a.a_1a_2\ldots a_n = a.a_1a_2\ldots(a_n - 1)\overline{9},$$

where $a_i \in \{0,\ldots,9\}$, $a_n \neq 0$ and $a \in \mathbb{Z}$. For instance

$$3.14 = 3.13\overline{9} \quad \text{and} \quad -3.14 = -3.13\overline{9}.$$

Why is $0.999\ldots = 1$? 5

- We see that every finite decimal expansion can also be written as an infinite decimal expansion.
- There is only exception one, namely 0.
- One way to understand why 0 is exceptional is because for positive numbers, the infinite expansion “looks” smaller, wile for negative numbers, the infinite expansion “looks” bigger. So it is not surprising that 0 is a singular case.

Decimal expansion 1

Theorem

A number is rational if and only if the decimal expansion is finite or repeating.

- Proof: We will for simplicity assume that $0 < x < 1$.
  $\Longleftarrow$: Assume first that x has a finite decimal expansion. Then
  $$x = 0.a_1 a_2 \ldots a_n = \frac{a_1 a_2 \ldots a_n}{10^n},$$
  which is a fraction of integers.

Decimal expansion 2

- Assume now that x has a repeating decimal expansion. We multiply by a power of 10 so that the decimal expansion of $10^s x$ starts repeating right after the decimal point. As an example consider $x = 0.1\overline{23}$, where $10x = 1.\overline{23}$. Then we shift one period to the left and subtract to cancel the infinite string of decimals

$$\begin{aligned} 10^2(10x) &= 123.\overline{23} \\ (10^2 - 1)10x &= 123.\overline{23} - 1.\overline{23} = 122 \\ x &= \frac{122}{(10^2 - 1)10} = \frac{122}{990}. \end{aligned}$$

Decimal expansion 3

- In general,

$$
\begin{aligned}
10^s x &= a.\overline{a_1 ... a_r} \\
(10^r - 1)10^s x &= (10^r - 1)a + a_1 \dots a_r \\
x &= \frac{(10^r - 1)a + a_1 \dots a_r}{(10^r - 1)10^s}
\end{aligned}
$$

  which is rational.
- $\Longrightarrow$: If $x = \frac{m}{n}$ the division will either terminate, or we will get repeating remainders after at most $n - 1$ steps.

Decimal expansion 4

- Notice that

$$0.a_1...a_r = \frac{a_1...a_r}{10^r} = \frac{a_1...a_r}{10\cdots0}, \quad \text{so} \quad 0.3 = \frac{3}{10},$$
$$0.\overline{a_1...a_r} = \frac{a_1...a_r}{10^r - 1} = \frac{a_1...a_r}{9\cdots9}, \quad \text{so} \quad 0.\overline{3} = \frac{3}{9}.$$

- We will call 1 followed by a string of r zeros a 10-block of length r and a string of r nines a 9-block of length r.

Decimal expansion 5

- As an example, consider $1/7$.

$$
\begin{array}{lll}
1 : 7 = 0.142857\ldots & & \\
\underline{-0} & & \\
10 & & \text{remainder } 1 \\
\underline{-7} & & \\
\quad 30 & & \text{remainder } 3 \\
\underline{-28} & & \\
\quad\; 20 & & \text{remainder } 2 \\
\underline{-14} & & \\
\quad\;\; 60 & & \text{remainder } 6 \\
\underline{-56} & & \\
\quad\;\;\; 40 & & \text{remainder } 4 \\
\underline{-35} & & \\
\quad\;\;\;\; 50 & & \text{remainder } 5 \\
\underline{-49} & & \\
\quad\;\;\;\;\;\; 1 & & \text{remainder } 1
\end{array}
$$

Decimal expansion 6

- There is a pattern in the decimal expansion of $1/7$, which can be explained as follows.

$$\frac{1}{7} = \frac{14}{98} = \frac{14}{100-2} = \frac{0.14}{1-2/10^2} = \sum_{k=0}^{\infty} 0.14(2/10^2)^k$$
$$= 0.14 + 0.0028 + 0.000056 + 0.00000112 + \cdots$$
$$= 0.\overline{142857}.$$

Historical remarks 1

- We have seen that a rational number has a decimal expansion that is either terminating or repeating, but how can we tell which type $1/n$ has, and what the length of the minimal repeating block is?
- Leibniz (1677) claimed that if n and 10 are relatively prime, then the decimal expansion of $1/n$ is periodic and the length of the repeating block is a factor in $n-1$.
- However, in 1685 Wallis pointed out that this fails for $1/21 = 0.\overline{047619}$, since 6 does not divide 20. Lambert claimed in 1758 that Leibniz's claim was true if n is prime, but he was not able to prove it.
- Lambert finally managed to prove it in 1769, using Fermat's Little Theorem, which shows that $10^{p-1} - 1$ is divisible by p if p is prime. This shows that $1/p$ can be extended to a fraction with a denominator equal to $10^{p-1} - 1$. This was independently rediscovered by Bernoulli in 1771.

Historical remarks 2

- The result can be further strengthened by using Euler's Theorem, which shows that if n and 10 are relatively prime, then $10^{\phi(n)} - 1$ is divisible by n, where $\phi(n)$ is the number of $1 \leq k \leq n$ that are relatively prime to n.
- It follows that if n is relatively prime to 10, then $1/n$ can be extended to a fraction with a denominator equal to $10^{\phi(n)} - 1$, and we can conclude that the length of the repeating block is a factor of $\phi(n)$. For example, the length of the repeating block of $1/21 = 0.\overline{047619}$ is 6, which divides $\phi(21) = 12$.
- This example shows that in order to solve a simple problem about decimal expansion, it was necessary to introduce techniques from number theory and modular arithmetic.
- We will therefore now take a break, and study some number theory before we continue with the decimal expansions.

Euler's Theorem and decimal expansion 1

- We define Euler's ϕ-function, $\phi(n)$, to be the number of natural numbers $1 \leq k \leq n$ with k and n relatively prime.
- Euler's Theorem says that if $\gcd(n, 10) = 1$, then $10^{\phi(n)} \equiv 1 \pmod{n}$, which is the same as saying that $n | (10^{\phi(n)} - 1)$, i.e., n divides a 9-block of length $\phi(n)$.
- From the decimal expansion of $1/7$, we see that

$$\begin{aligned}(10^6 - 1)1/7 &= 142857.142857\ldots - 0.142857\ldots \\ &= 142857,\end{aligned}$$

so that $999999 = 7 \cdot 142857$. This shows that 7 divides a 9-block, $10^6 - 1$, of length equal to the period of $1/7$ and that $(10^6 - 1)/7$ is the repeating block. Since $\phi(7) = 6$, this agrees with Euler's Theorem.

Euler's Theorem and decimal expansion 2

- If $\gcd(n, 10) = 1$, then the order of 10 in $\mathbb{Z}_n$ is the smallest positive number k such that n divides $10^k - 1$. We know that the order is a factor of $\phi(n)$.
- We also know that the order is equal to the length of the minimal repeating block.
- If n divides $10^l - 1$, then there is a repeating block of length l, but if $k < l$, then this block consist of several copies of the minimal repeating block, so $k|l$.

Types of rational decimal expansion 1

- Consider $\frac{m}{n}$ where $0 < m < n$ and $\gcd(m,n) = 1$, so that $0 < m/n < 1$.

Terminating	$0.d_1 \dots d_t (d_t \neq 0)$	$\frac{m}{2^u 5^v}$, $t = \max(u,v)$	$\frac{M_t}{10^t} = \frac{d_1 \dots d_t}{10^t}$
Simple-periodic	$0.\overline{d_1 \dots d_r}$	$\frac{m}{n}$, $(n,10) = 1$	$\frac{M_s}{10^r - 1} = \frac{d_1 \dots d_r}{10^r - 1}$
Delayed-periodic	$0.d_1 \dots d_t \overline{d_{t+1} \dots d_{t+r}}$	$\frac{m}{n_1 n_2}$, $n_1 = 2^u 5^v$, $\gcd(n_2, 10) = 1$, $t = \max(u,v) > 1$, $n_2 > 1$.	$\frac{M_d}{10^t(10^r - 1)}$

- Since $M_d < 10^t(10^r - 1)$, we can divide by $(10^r - 1)$ to get a quotient with at most t digits.

$$M_d = (10^r - 1) d_1 \dots d_t + d_{t+1} \dots d_{t+r} \quad \text{and}$$

$$\frac{M_d}{10^t(10^r - 1)} = \frac{d_1 \dots d_t}{10^t} + \frac{d_{t+1} \dots d_{t+r}}{10^t(10^r - 1)}.$$

Types of rational decimal expansion 2

- Notice that

$$M_d = (10^r - 1)d_1 \dots d_t + d_{t+1} \dots d_{t+r},$$

which we can also write as

$$\begin{aligned} 10^r d_1 \dots d_t + d_{t+1} \dots d_{t+r} - d_1 \dots d_t \\ = d_1 \dots d_{t+r} - d_1 \dots d_t. \end{aligned}$$

- This shows how to convert between M_d and the d_i. We have shown that $x = 0.1\overline{23} = 122/990$, and we can write $M_d = 122$ as either $99 \cdot 1 + 23$ or as $123 - 1$.

Types of rational decimal expansion 3

- Proof: m/n is terminating if and only if

$$m/n = \frac{m}{2^u 5^v} = \frac{M_t}{10^t}.$$

- m/n is simple-periodic if and only if we can cancel the decimals by shifting one period, i.e.

$$(10^r - 1)m/n = M_s.$$

- m/n is delayed-periodic if and only if we can cancel the decimals by shifting one period and moving the period t places, i.e

$$10^t(10^r - 1)m/n = M_d. \quad \square$$

-

$$0.\overline{062} = 62/999, \quad 0.0\overline{62} = 62/(10 \cdot 99) = 62/990.$$

- Notice that there may be initial 0's in the d_i's, that the type of expansion only depends on n and not on m, and that the fractions in the last column need not be reduced.

Types of rational decimal expansion 4

- In the simple-periodic case, the repeating block is simply $m(10^r - 1)/n$, but in the delayed-periodic case, we must divide $m10^t(10^r - 1)/n$ by $10^r - 1$ to separate the finite and repeating parts. However, it is easier to divide $m10^t/n_1$ by n_2 to keep the numbers smaller, as the following examples show.
- $$\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{5\cdot 3}{10\cdot 9} = \frac{15}{10\cdot 9} = \frac{1\cdot 9+6}{10\cdot 9} = \frac{1}{10} + \frac{6}{10\cdot 9} = 0.1\overline{6},$$
$$\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{5}{10\cdot 3} = \frac{1\cdot 3+2}{10\cdot 3} = \frac{1}{10} + \frac{2}{10\cdot 3} = \frac{1}{10} + \frac{6}{10\cdot 9} = 0.1\overline{6}.$$
- $$\frac{1}{24} = \frac{1}{2^3\cdot 3} = \frac{5^3\cdot 3}{10^3\cdot 9} = \frac{375}{10^3\cdot 9} = \frac{41\cdot 9+6}{10^3\cdot 9} = \frac{41}{10^3} + \frac{6}{10^3\cdot 9} = 0.041\overline{6},$$
$$\frac{1}{24} = \frac{1}{2^3\cdot 3} = \frac{5^3}{10^3\cdot 3} = \frac{125}{10^3\cdot 3} = \frac{41\cdot 3+2}{10^3\cdot 3} = \frac{41}{10^3} + \frac{2}{10^3\cdot 3} = 0.041\overline{6}.$$

Types of rational decimal expansion 5

- Notice that $10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 = 76923 \cdot 13 = 142857 \cdot 7$.

$$\frac{1}{26} = \frac{1}{2 \cdot 13} = \frac{5 \cdot 76923}{10 \cdot (10^6 - 1)} = \frac{384615}{10 \cdot (10^6 - 1)} = 0.0\overline{384615}.$$

-

$$\frac{1}{28} = \frac{1}{2^2 \cdot 7} = \frac{25}{10^2 \cdot 7} = \frac{3 \cdot 7 + 4}{10^2 \cdot 7} = \frac{3}{10^2} + \frac{4}{10^2 \cdot 7}$$
$$= \frac{3}{10^2} + \frac{4 \cdot 142857}{10^2 \cdot (10^6 - 1)} = \frac{3}{10^2} + \frac{571428}{10^2 \cdot (10^6 - 1)} = 0.03\overline{571428},$$
$$\frac{1}{28} = \frac{1}{2^2 \cdot 7} = \frac{25 \cdot 142857}{10^2 \cdot (10^6 - 1)} = \frac{3 \cdot (10^6 - 1) + 571428}{10^2 \cdot (10^6 - 1)}$$
$$= \frac{3}{10^2} + \frac{571428}{10^2 \cdot (10^6 - 1)} = 0.03\overline{571428}.$$

- Notice how the type of the decimal expansion of m/n and the size of r and t only depend on n.

Factoring $10^n - 1$

- To find denominators with short periods, we use the following table. The period of $1/p$ is the r for which p first appears as a factor in $10^r - 1$. Notice how $3, 11$ and 13 appear earlier than given by Euler's Theorem, while 7 first appears in $10^6 - 1$.

$$
\begin{aligned}
10^1 - 1 &= 3^2 \\
10^2 - 1 &= 3^2 \cdot 11 \\
10^3 - 1 &= 3^3 \cdot 37 \\
10^4 - 1 &= 3^2 \cdot 11 \cdot 101 \\
10^5 - 1 &= 3^2 \cdot 41 \cdot 271 \\
10^6 - 1 &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \\
10^7 - 1 &= 3^2 \cdot 239 \cdot 4649 \\
10^8 - 1 &= 3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137 \\
10^9 - 1 &= 3^4 \cdot 37 \cdot 333667 \\
10^{10} - 1 &= 3^2 \cdot 11 \cdot 41 \cdot 271 \cdot 9091 \\
10^{11} - 1 &= 3^2 \cdot 21649 \cdot 513239 \\
10^{12} - 1 &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901
\end{aligned}
$$

$1/p$ for p prime

- If p is prime other than 2 or 5, then it follows from Fermat's Theorem that p divides a 9-block of length $p-1$, and therefore the order, r, divides $p-1$.
- However, for composite numbers, n, it is not necessarily true that n divides a 9-block of length $n-1$. Since

  $$10^{20}-1=3^2\cdot 11\cdot 41\cdot 101\cdot 271\cdot 3541\cdot 9091\cdot 27961,$$

  we see that it fails for 21. It also shows that the order, r, does not necessarily divide $n-1$. In fact, $\phi(21)=\phi(7)\phi(3)=6\cdot 2=12$, and the order, 6, divides $\phi(n)$ and not n.

Summary of $1/n$

- t is the length of the terminating part and r is the length of the repeating block in the decimal expansion of $1/n$.

$1/n$	t	r	$\phi(n)$	$1/n$	t	r	$\phi(n)$
$1/2 = 0.5$	1			$1/22 = 0.0\overline{45}$	1	2	10
$1/3 = 0.\overline{3}$		1	2	$1/23 = 0.\overline{0434782608695652173913}$		22	22
$1/4 = 0.25$	2			$1/24 = 0.041\overline{6}$	3	1	8
$1/5 = 0.2$	1			$1/25 = 0.04$	2		
$1/6 = 0.1\overline{6}$	1	1	2	$1/26 = 0.0\overline{384615}$	1	6	12
$1/7 = 0.\overline{142857}$		6	6	$1/27 = 0.\overline{037}$		3	18
$1/8 = 0.125$	3			$1/28 = 0.03\overline{571428}$	2	6	12
$1/9 = 0.\overline{1}$		1	6	$1/29 = 0.\overline{0344827586206896551724137931}$		28	28
$1/10 = 0.1$	1			$1/30 = 0.0\overline{3}$	1	1	8
$1/11 = 0.\overline{09}$		2	10	$1/31 = 0.\overline{032258064516129}$		15	30
$1/12 = 0.08\overline{3}$	2	1	4	$1/32 = 0.03125$	5		
$1/13 = 0.\overline{076923}$		6	12	$1/33 = 0.\overline{03}$		2	20
$1/14 = 0.0\overline{714285}$	1	6	6	$1/34 = 0.0\overline{2941176470588235}$	1	16	16
$1/15 = 0.0\overline{6}$	1	1	8	$1/35 = 0.0\overline{285714}$	1	6	24
$1/16 = 0.0625$	4			$1/36 = 0.02\overline{7}$	2	1	12
$1/17 = 0.\overline{0588235294117647}$		16	16	$1/37 = 0.\overline{027}$		3	36
$1/18 = 0.0\overline{5}$	1	1	6	$1/38 = 0.0\overline{263157894736842105}$	1	18	18
$1/19 = 0.\overline{052631578947368421}$		18	18	$1/39 = 0.\overline{025641}$		6	24
$1/20 = 0.05$	2			$1/40 = 0.025$	3		
$1/21 = 0.\overline{047619}$		6	12	$1/41 = 0.\overline{02439}$		5	40

- What can you say about $1/27$ and $1/37$? Why?
- $10^3 - 1 = 3^3 \cdot 37$.

Primes with given period

- Primes p with repeating decimal expansions of period r in $1/p$.

Period	Primes
1	3
2	11
3	37
4	101
5	$41, 271$
6	$7, 13$
7	$239, 4649$
8	$73, 137$
9	333667
10	9091
11	$21649, 513239$
12	9901
13	$53, 79, 265371653$
14	909091
15	$31, 2906161$
16	$17, 5882353$
17	$2071723, 5363222357$
18	$19, 52579$
19	1111111111111111111
20	$3541, 27961$

- Notice how $7, 17$ and 19 have maximal periods, $p - 1$. Gauss conjectured in 1801 that there are infinitely many primes with maximal periods, but this has not been proved.

Periods of inverse primes

- Here are the periods of $1/p$ for all primes less than 101 except for 2 and 5.

p	r	p	r	p	r
3	1	31	15	67	33
7	6	37	3	71	35
11	2	41	5	73	8
13	6	43	21	79	13
17	16	47	46	83	41
19	18	53	13	89	44
23	22	59	58	97	96
29	28	61	60	101	4

Periods of $1/n$ (optional)

- If $n = p^k$, the period is a divisor of $\phi(p^k) = (p-1)p^{k-1}$, but there is no simple formula.
- If $n = n_1 n_2$ where $\gcd(n_1, n_2) = 1$, then it can be shown that the period of $1/n$ is the least common multiple of the periods of $1/n_1$ and $1/n_2$.

Midy's Theorem (optional)

- If a/p is a reduced fraction with p prime and the period of a/p is $2n$, i.e.,

$$\frac{a}{p} = 0.\overline{a_1 \ldots a_n a_{n+1} \ldots a_{2n}}$$

then it can be shown the digits in the second half of the repeating decimal period are the 9s complement of the corresponding digits in its first half. In other words, $a_i + a_{i+n} = 9$. For example,

$$\frac{1}{13} = \overline{0.076923} \quad \text{with } 076 + 923 = 999, \text{ and}$$

$$\frac{1}{17} = \overline{0.0588235294117647} \quad \text{with}$$

$$05882352 + 94117647 = 99999999.$$

Cyclic numbers (optional)

- Consider the following decimal expansions

$$1/7 = 0.\overline{142857}$$
$$2/7 = 0.\overline{285714}$$
$$3/7 = 0.\overline{428571}$$
$$4/7 = 0.\overline{571428}$$
$$5/7 = 0.\overline{714285}$$
$$6/7 = 0.\overline{857142}$$

- Notice how the digits of the repeating blocks are cyclic permutations of each other.

Multiple cycles (optional)

- Sometimes the numbers m/n break into several cycles. For example, the multiples of 1/13 can be divided into two sets:

$$\begin{aligned}
1/13 &= 0.\overline{076923} \\
10/13 &= 0.\overline{769230} \\
9/13 &= 0.\overline{692307} \\
12/13 &= 0.\overline{923076} \\
3/13 &= 0.\overline{230769} \\
4/13 &= 0.\overline{307692}
\end{aligned}$$

where each repeating block is a cyclic re-arrangement of 076923 and

$$\begin{aligned}
2/13 &= 0.\overline{153846} \\
7/13 &= 0.\overline{538461} \\
5/13 &= 0.\overline{384615} \\
11/13 &= 0.\overline{846153} \\
6/13 &= 0.\overline{461538} \\
8/13 &= 0.\overline{615384}
\end{aligned}$$

where each repeating block is a cyclic re-arrangement of 153846.