UiO : University of Oslo

Mean and Extreme

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- There should be a line of sight back towards school mathematics.

Arithmetic series

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- In an arithmetic series, every term is the arithmetic mean of the two surrounding terms.

$$\frac{1}{2}(a_{n+1} + a_{n-1}) = \frac{1}{2}(a_n + d + a_n - d) = \frac{1}{2}2a_n = a_n.$$

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$$(a_{n+1}a_{n-1})^{1/2} = [(a_n r)(a_n/r)]^{1/2} = (a_n^2)^{1/2} = a_n.$$

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$$\frac{2}{\frac{1}{1/(n+1)} + \frac{1}{1/(n-1)}} = \frac{2}{n+1+n-1} = \frac{2}{2n} = \frac{1}{n}.$$

Confusing means — Simpson's paradox

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- In 1973 UC Berkeley admitted 44% of males and 35% of females who applied to grad school. The tables show admission data from the six largest departments.

Department	Male acceptance rate	Female acceptance rate
A	62%	82%
B	63%	68%
C	37%	34%
D	33%	35%
E	28%	24%
F	6%	7%

Department	Male		Female	
	Applicants	%	Applicants	%
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	373	6%	341	7%

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- The arithmetic mean may look innocent, but can be devious.

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- $z = \sqrt{xy} = \mathrm{GM}(x, y)$.

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- Assume that the distance is d. Then your average speed was

$$\frac{2d}{d/30 + d/60} = \frac{2}{1/30 + 1/60} = \frac{2 \cdot 60}{2 + 1} = 40 = H(30, 60).$$

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- The term harmonic is related to music theory.

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$$
\begin{aligned}
&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots \\
>\ &1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\cdots
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- This argument shows that

$$\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{k}{2},$$

and we see that the series diverges.

The AGH inequality

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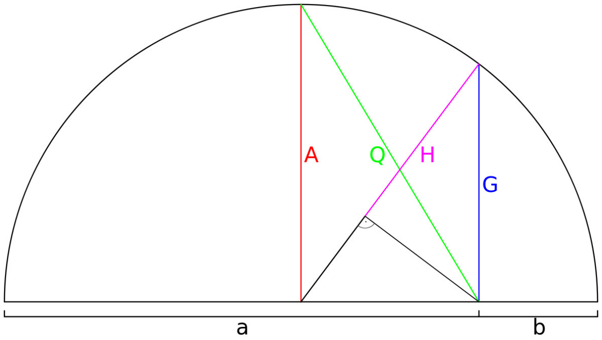

$$\mathrm{AM}(a,b) \geq \mathrm{GM}(a,b) \geq \mathrm{HM}(a,b).$$

Proof of the AGH inequality

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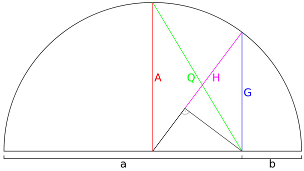

$$\left(\frac{a+b}{2}\right)^2 = G^2 + \left(a - \frac{a+b}{2}\right)^2$$

$$\frac{(a+b)^2}{4} = G^2 + \frac{(a-b)^2}{4}$$

$$(a+b)^2 = 4G^2 + (a-b)^2$$

$$2ab = 4G^2 - 2ab$$

$$G^2 = ab$$

$$G = \sqrt{ab} = \mathrm{GM}(a,b).$$

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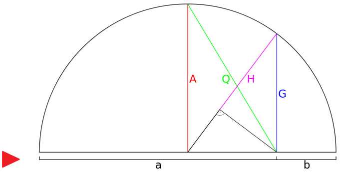

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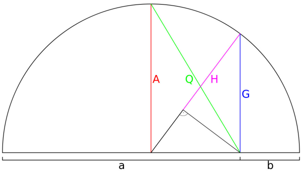
- 

- By similar triangles $A/G = G/H$ or $G^2 = AH$. Hence

$$H = \frac{G^2}{A} = \frac{ab}{\frac{a+b}{2}} = \frac{2}{\frac{a+b}{ab}} = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \mathrm{HM}(a, b).$$

Maximum volume of a cut off box

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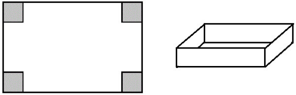

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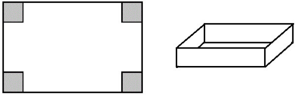
- 
- Consider a rectangle of width 1 and length L. We cut off squares of side x in each corner and fold to get a box of volume
$$V(L, x) = x(L - 2x)(1 - 2x).$$

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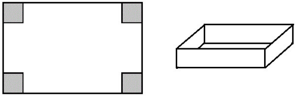
- 
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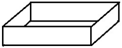
-  
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$$x = \frac{L + 1 - \sqrt{(L+1)^2 - 3L}}{6}.$$
- If we set $L = 1$, we get
$x = \frac{1+1-\sqrt{(1+1)^2-3\cdot 1}}{6} = \frac{2-\sqrt{2^2-3}}{6} = \frac{1}{6}.$

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- Let $A(t)$ be the area of the region inside $W(t)$, let $P(t)$ be the perimeter of $W(t)$ and let $V(t)$ be the volume of the box.
- I claim that $A'(t) = -P(t)$.

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- We have $A'(t) = \lim_{h \to 0} \frac{A(t+h)-A(t)}{h}$, and we can interpret $A(t+h) - A(t)$ as the negative of the area of a "ring" of thickness h. Since the area of the ring will have area approximately equal to $P(t)t$, we get that

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- We have $V(t) = A(t)t$, so $V'(t) = A'(t)t + A(t) = -P(t)t + A(t) = 0$ precisely when the area of the base equals the area of the wall.

Source of counterexamples

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- Note that
$$\lim_{x \to \infty} x \sin(1/x) = \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{y \to 0} \frac{\sin(y)}{y} = 1.$$

$\sin(1/x)$

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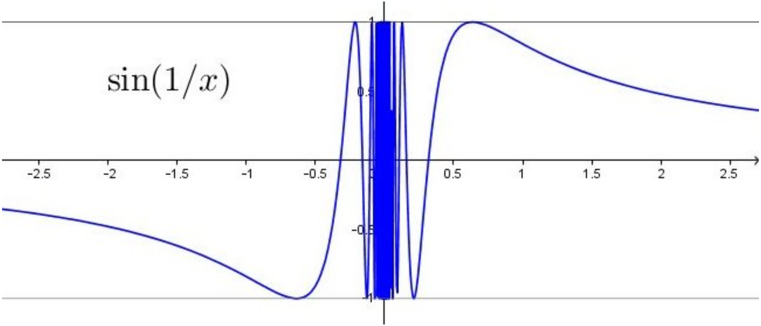
- $$f_0(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



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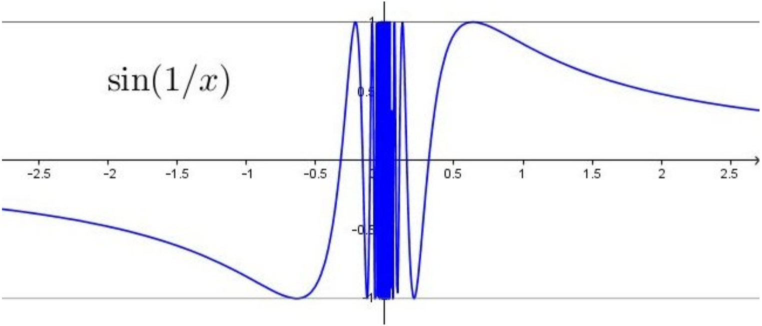

- f_0 is not continuous at $x = 0$, since $\lim_{x \to 0} f_0(x)$ does not exist.

$x \sin(1/x)$

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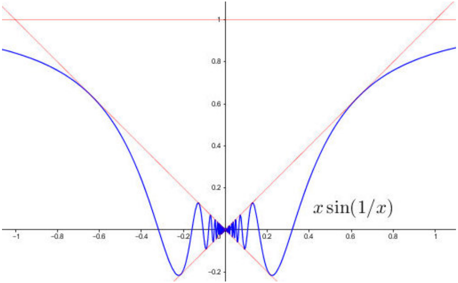

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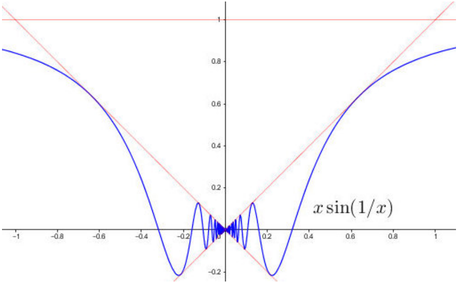

- Remember that $\lim_{x\to\infty} f_1(x) = 1$.

$x \sin(1/x)$ part 2

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- f_1 is continuous, since it is squeezed by $\pm x$, but

$$\lim_{x\to 0}\frac{f_1(x)-f_1(0)}{x-0}=\lim_{x\to 0}\frac{x\sin(1/x)-0}{x-0}=\lim_{x\to 0}\sin(1/x),$$

does not exist, so f_1 is not differentiable at $x=0$.

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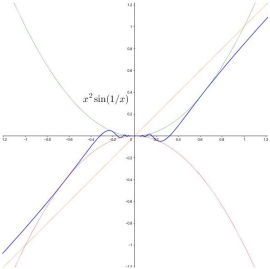

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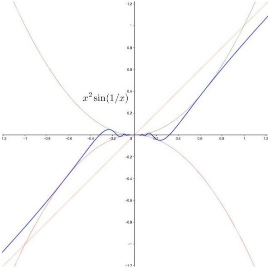

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- Setting $y = 1/x$ and using L'Hôpital's rule, we get
$$\lim_{x\to\infty}(x^2 \sin(1/x) - x) = \lim_{y\to 0}(\sin y/y^2 - 1/y) =$$
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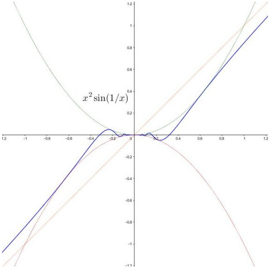

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- f_2 is differentiable, since it is squeezed by $\pm x^2$.

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- $$f_2'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x \sin(1/x) = 0.$$

  However, for $x \neq 0$ we have $f_2'(x) = 2x \sin(1/x) - \cos(1/x)$, and

  $$\lim_{x\to 0} f_2'(x) = \lim_{x\to 0} (2x \sin(1/x) - \cos(1/x))$$

  does not exist.

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- So f_2 is differentiable, but not continuously differentiable!
- This is the mother of all counterexamples!

Monotonicity

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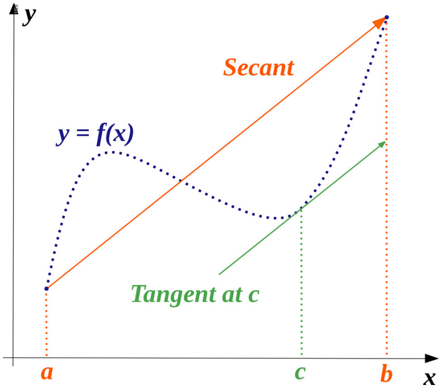
- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



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- f is strictly increasing if $x < y \implies f(x) < f(y)$.
- Assume that $f' > 0$ on (a, b). Given $a < x < y < b$, we can find $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) > 0$. It follows that

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- Limits do not preserve strict inequalities.

Extreme point

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- Assume that c is a minimum point and that $f'(c)$ exists. Consider $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$. If h is positive, the fraction is positive, and if h is negative, the fraction is negative. Since the limit exists, it must be zero.

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- Assume that f' exists around c, and $f'(x)$ is positive for $x > c$ and negative for $x < c$. If $x > c$, then there is a d between c and x such that $f(x) - f(c) = f'(d)(x - c) > 0$. If $x < c$, then there is a d between x and c such that $f(c) - f(x) = f'(d)(c - x) < 0$. It follows that c is a minimum point.

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- However, the converse is not always true.

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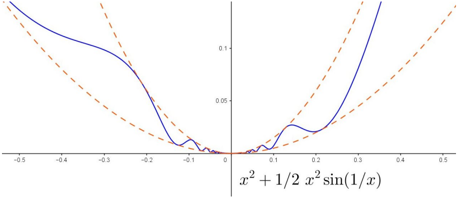
- Since $x^2 + x^2 \sin(1/x))$ has infinitely many zeros, we instead start with x^2 and use $f(x) = x^2(+1/2 \sin(1/x))$, which satisfies $1/2\, x^2 \geq f(x) \geq 3/2\, x^2$.



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- We have $f'(x) = 4x + 2x \sin(1/x) - \cos(1/x)$, and if x is close to zero, the first two terms will be close to zero, too, while the last term will oscillate between 1 and -1.

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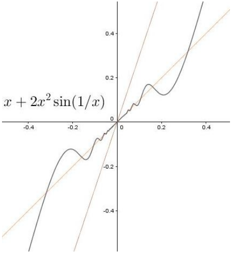
- Then $f'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x)$, and when x is close to zero, then will oscillate between 3 and -1, so f' will be both positive and negative in every neighborhood of 0.



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- Let us consider some examples.
- $f(x) = x^3$ has $f'(0) = 0$, but 0 is not an extremum, but a point of inflection.
- $f(x) = x^3 + x$ shows that f' does not have to be 0 at a point of inflection.

Point of inflection 2

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- $f(x) = x^{1/3}$ has a point of inflection at 0, has a tangent line at 0, but $f'(0)$ and $f''(0)$ do not exist. (Vertical tangent line. Just bend a bit, and both derivatives will exist.)

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- $$f(x) = \begin{cases} x^2 + x \text{ if } x \geq 0, \\ -x^2 - 2x \text{ if } x < 0 \end{cases}$$

  does not have a tangent line at 0, since the first derivatives do not match. However, the second derivative changes sign at 0. Is this a point of inflection? I have chosen to not include this, but some people do.

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Point of inflection 4

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- Proof of 3: We use MVT go get x_1 between c and x with

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- Combining this, we get
$$\begin{aligned} f(x) &= f(c)+f'(x_1)(x-c) \\ &= f(c)+f'(c)(x-c)+f''(x_2)(x-c)(x_1-c). \end{aligned}$$

Point of inflection 5

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Point of inflection 6

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in a neighborhood of 0, so 0 is an isolated minimum of $f'(x)$. We have $f''(0) = 0$, but $f''(x) = 6x - \sin(1/x) - 6x \cos(1/x) + 12x^2 \sin(1/x)$ does not change sign.

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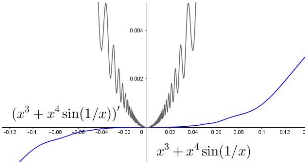
- 

Point of inflection 7

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- The first two terms give us the shape we want, and the last terms is so small that we can ignore it.

Point of inflection 8

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- Converse to 3 is false:
  $f(x) = x^3 + 1/2\, x^3 \sin(1/x) = x^3(1 + 1/2\ \sin(1/x))$ lies below the tangent ($y = 0$) on one side and above the tangent on another, but
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