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Numbers

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Natural numbers

- Let $\mathbb{N}$ denote the positive, natural numbers $\{1, 2, 3, \ldots\}$.
- Computer scientists start with 0, and many mathematicians now include 0 in $\mathbb{N}$.
- Remember that positive means > 0 and negative means < 0, so nonnegative is not the same as positive, but means positive or zero.

Why is $(-1)(-1) = 1$?

- One way to understand why $(-1)(-1) = 1$, is to say that multiplying by -1 is the same as “flipping” across zero on the number line, in which case, flipping twice does nothing.
- However, it is instructive to also see an algebraic proof. Assume that we know how to multiply natural numbers, and that we want to extend this to integers. We want to do this in such a way that the following three properties are preserved.
  1. Commutative $ab = ba$
  2. Associative $(ab)c = a(bc)$
  3. Distributive $a(b + c) = ab + ac$

Why is $(-1)(-1) = 1$? 1

- Assume that $a, b \in \mathbb{N}$. We know that

$$a(-b) = (-b) + (-b) + \cdots + (-b) = -ab \tag{1}$$

by repeated addition.

- To compute $(-a)b$, we use commutativity and Equation (1) to get

$$(-a)b = b(-a) = -ba = -ab.$$

Why is $(-1)(-1) = 1$? 2

- We want to show that $(-1)(-1) = 1$, and to do that, we want to show that $(-1)(-1)$ "behaves" like 1.
- We consider $(-1)(-1) - 1$ and use distributivity

$$
\begin{aligned}
(-1)(-1) - 1 &= \\
(-1)(-1) + (-1) &= \\
(-1)(-1) + (-1) \cdot 1 &= \\
(-1)(-1 + 1) &= \\
(-1) \cdot 0 &= 0.
\end{aligned}
$$

Hence $(-1)(-1) = 1$.

Why do we use parentheses?

- We use a parenthesis when we need to manipulate order of operations. $2 \cdot 3 + 4$ means first multiply and then add. If we want to first add, we must write $2(3 + 4)$.
- We can write $1 - (-2 + x) = 1 + (-1)(-2 + x) = 1 + (2 - x) = 1 + 2 - x$, so we see that we must change the sign of all terms when we remove a parenthesis with a minus in front of it.
- Notice that wrote $1 - (-2 + x)$ and not for example $1 - (-2 + 3)$. If there were just numbers, we would probably evaluate the term inside the parenthesis instead of keeping the parenthesis.
- You can also explain the rule by giving an accounting example. You have \$10 and spend \$1 and \$2. You can either write $10 - 1 - 2$ or $10 - (1 + 2)$, where you first compute as you spend, and in the second example add up all your expenses.

Rounding 1

- Why do we round 0.5 to 1 and not 0?
- This is a convention, but it does have one advantage and one disadvantage.
- If you are doing a long computation, and you have found that the first digit is 0.5, the you know that it should be rounded to 1. If your convention is that 0.5 goes to 0, while 0.5 followed by anything other than just 0's goes to 1, then you need to keep computing.

Rounding 2

- However, always rounding 0.5 up creates a bias away from 0. What this means is that numbers tend to become larger. If you add two numbers that are 0.5 and 1.5, you would first round them to 1 and 2 and get 3 as the sum, while the actual sum is 2.
- For this reason, some people use round-to-even, which means that 0.5 is rounded to 0, while 1.5 is rounded to 2. If you then add two numbers that are 0.5 and 1.5, you get $0 + 2 = 2$, which is the right answer, as opposed to 3, which we got above.

Division

- There are two ways to interpret division.
- Partitive division: You and your friend have 6 apples. If you share them, you get $6 : 2 = 3$ apples each. Thing/number=thing.
- Measurement division: You have a barrel of 6l of water, and glasses that take 0.5l. You can fill $6 : 0.5 = 12$ glasses. Thing/thing=number.
- Notice that measurement division also makes sense when the divisor is not an integer.

Division by zero

- The key to understanding division and fractions is that

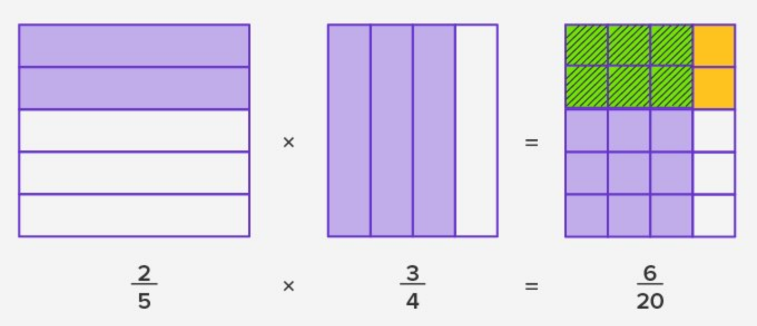
$$\frac{a}{b} = c \iff a = bc. \tag{2}$$

  This shows why we cannot divide by 0. If $b = 0$, we get

$$\frac{a}{0} = c \iff a = 0 \cdot c = 0,$$

  which shows that we get a contradiction if we try to assign a value to $a/0$ when $a \neq 0$.
- But what if $a = 0$? In that case, the above equation just says that $0 = 0 \cdot c = 0$, which is true for any c. But that is precisely the problem. We could theoretically define $0/0$ to be anything, without violating (2), but which value should we choose? Since we theoretically could pick any value, we say that $0/0$ is an indeterminate form.

Multiplication of fractions

- We can interpret multiplication of fractions geometrically.
- 

Multiplication of fractions 1

- We will not give a thorough discussion of fractions, just give a brief outline of how it could be developed.
- We think of the unit fraction $1/n$, where $n \in \mathbb{N}$, as one n-part of the interval $[0, 1]$.
- We think of the fraction m/n where $m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}$ as m n-parts. Notice how the term numerator denotes the number of things we have, and the denominator describes what we have m of.
- Then $k(m/n) = (km)/n$.
- We think of the product of a unit fraction $1/l$ and a fraction m/n as $m/(ln)$, i.e., m ln-parts.

Multiplication of fractions 2

- We then have

$$\left(\frac{k}{l}\right)\left(\frac{m}{n}\right) = \left(k\left(\frac{1}{l}\right)\right)\left(\frac{m}{n}\right) = k\left(\left(\frac{1}{l}\right)\left(\frac{m}{n}\right)\right)$$
$$= k\left(\frac{m}{ln}\right) = \frac{km}{ln}.$$

- Notice how the crucial point was to first consider unit fractions.

Division by fractions

- Many students do not understand why dividing by a fraction is the same as inverting the second fraction and multiplying

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b}\frac{d}{c}. \tag{3}$$

To see this, we must show that if multiply the number on the right by the divisor, we get the dividend, i.e.,

$$\left(\frac{a}{b}\frac{d}{c}\right)\frac{c}{d} = \frac{a}{b}\left(\frac{d}{c}\frac{c}{d}\right) = \frac{a}{b}.$$

Division by fractions 1

- Another way to see this is to use complex fractions

$$\frac{\frac{a}{b}bd}{\frac{c}{d}bd} = \frac{ad}{bc}.$$

Division by fractions 2

- It is also instructive to consider unit fractions, $1/d$. There are many ways to argue that

$$a : \frac{1}{d} = ad.$$

- Then

$$\frac{a}{b} : \frac{1}{d} = \frac{1}{b} a : \frac{1}{d} = \frac{1}{b} ad = \frac{ad}{b}.$$

- It then follows from associativity that

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} : \left(c \frac{1}{d} \right) = \left(\frac{a}{b} : c \right) : \frac{1}{d} = \frac{ad}{bc}.$$

Powers

- Assume that we have defined a^n with $n \in \mathbb{N}$ to be

$$a^n = \overbrace{a \cdot \ldots \cdot a}^{n}. \tag{4}$$

  For n, $m \in \mathbb{N}$ it is easy to see that we have the following property

$$a^n a^m = \overbrace{a \cdots a}^{n} \overbrace{a \cdots a}^{m} = \overbrace{a \cdots a}^{n+m} = a^{n+m}. \tag{5}$$

- We now want to extend Definition (4) to $n \in \mathbb{Z}$ in way that preserves property (5). In other words, we will assume that (5) holds, and see what that implies about a^0 and a^{-n} for $n \in \mathbb{N}$.

Powers 1

- Setting $m = 0$ in (5), we get

$$a^n = a^{n+0} = a^n \cdot a^0,$$

so if $a \neq 0$, we can divide by a^n and conclude that $a^0 = 1$. (We will discuss 0^0 later.)

- We now set $m = -n$ in (5) and get

$$1 = a^0 = a^{n-n} = a^n a^{-n},$$

so it follows that

$$a^{-n} = \frac{1}{a^n}.$$

- You can also make a table showing that $10^3 = 1000$, $10^2 = 100$, $10^1 = 10$, and then ask students to spot the pattern and see guess that $10^0 = 1$ and $10^{-1} = 1/10$.

Empty Product

- Another way to understand this, is to interpret a^0 as an "empty product".
- 10^2x means that x is multiplied by 10 twice. 10^0x means that x is multiplied by 10 zero times. But not multiplying is the same as multiplying by 1, so $10^0x = x$, and 10^0 must be 1.
- In the same way, $2a = a + a$, while $0a$ is to add x zero times. But not adding at all is the same as adding 0, so $0a = 0$.
- The "empty sum" $0 \cdot a$ is the additive identity 0, while the "empty product" a^0 is the multiplicative identity 1.
- So $a^0 = 1$ for the same reason as $0a = 0$.

Why is $0! = 1$?

- Another "empty product" is $0!$. We have $n! = n(n-1)\cdots 2 \cdot 1$ for $n \in N$, but it is sometimes convenient to have an expression for $0!$, too.
- In principle we could define 0! to be whatever we want, but we want our definition to preserve the properties we like.
- We have $n! = n(n-1)!$ for $n \geq 2$, and if we want this to hold for $n = 1$, we must set $0! = 1$.
- Notice that we use this convention in for instance the formula
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Fractional Exponents

- Again we want to extend a definition to a larger set of numbers by preserving a property. We know that for m, $n \in \mathbb{N}$ we have

$$(a^n)^m = \underbrace{a^n \cdots a^n}_{m} = \underbrace{(\overbrace{a \cdots a}^{n}) \cdots (\overbrace{a \cdots a}^{n})}_{m} = \overbrace{a \cdots a}^{n \cdot m} = a^{n \cdot m}. \qquad (6)$$

  We want to extend the definition of a^n to $n \in \mathbb{Q}$, while maintaining property (6), so we write $x = a^{1/n}$.

- Then

$$x^n = (a^{1/n})^n = a^{\left(\frac{1}{n}n\right)} = a^1 = a$$

  so we see that

$$a^{1/n} = \sqrt[n]{a}.$$

- Using property (6) again, we get that

$$a^{m/n} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}.$$

Is $0^0 = 1$?

- We have seen that for $a \neq 0$ we have $a^0 = 1$, so $\lim_{a \to 0} a^0 = 1$. It therefore seems natural to define $0^0 = 1$. However, for $x > 0$, we have $0^x = 0$ and it follows that $\lim_{x \to 0^+} 0^x = 0$.
- This shows that the function $f(a, x) = a^x$ does not have a limit at $(0, 0)$ since we get different values depending on how we approach $(0, 0)$. It follows that f is not continuous at $(0, 0)$.
- That makes it harder to find a good value for 0^0, but not impossible.

Is $0^0 = 1$? 1

- We often write a polynomial as

$$p(x) = \sum_{k=0}^{n} a_k x^k.$$

- However, then

$$p(0) = a_0 0^0 + \cdots + a_n 0^n = a_0 0^0 = a_0,$$

and we are implicitly assuming that $0^0 = 1$.

Is $0^0 = 1$? 2

- We can also consider a power series like

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Then

$$f(0) = 1 = \sum_{n=0}^{\infty} 0^n = 0^0.$$

If we do not define 0^0 to be 1, we will have trouble with even simple expressions like this.

Is $0^0 = 1$? 3

- Another example is the Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

- Setting $a = 0$ on both sides and assuming $b \neq 0$ we get

$$b^n = (0+b)^n = \sum_{k=0}^{n} \binom{n}{k} 0^k b^{n-k} = \binom{n}{0} 0^0 b^n = 0^0 b^n,$$

  where, we have used that $0^k = 0$ for $k > 0$, and that $\binom{n}{0} = 1$.
- We see that we must set $0^0 = 1$ in order for the binomial theorem to be valid.

Is $0^0 = 1$? 4

- In order for the differentiation rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

  to hold for $n = 1$ when $x = 0$, we get

$$1 = \frac{d}{dx}x = \frac{d}{dx}x^1 = 1 \cdot x^{1-1} = x^0,$$

  which requires $0^0 = 1$.
- So to sum up, we must write $0^0 = 1$ to make many expressions work, and this also agrees with our earlier discussion of the "empty product".

Rational numbers

- We will study the rational numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

  We want to show that $\sqrt{2}$ is irrational.
- We will need the following lemma.

Lemma

A natural number a is even if and only if a^2 is even.

- Proof: If a is even we can write $a = 2k$ with $k \in \mathbb{Z}$ and then

$$a^2 = (2k)^2 = 4k^2 = 2(2k^2),$$

  so we see that

$$a \text{ is even} \implies a^2 \text{ is even.}$$

Rational numbers 1

- In order to show

$$a \text{ is even} \iff a^2 \text{ is even},$$

we will use that

$$p \implies q \quad \text{is the same as} \quad \neg p \iff \neg q.$$

- So we will show that

$$a \text{ is odd} \implies a^2 \text{ is odd}.$$

- If $a = 2k + 1$ with $k \in \mathbb{Z}$, then

$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

so a^2 is odd. $\square$

Rational numbers 2

- We can now prove that

Theorem

$\sqrt{2}$ is irrational.

- Proof: We will assume that $\sqrt{2}$ is rational and can be written as a/b, where $a, b \in \mathbb{Z}$ are relatively prime, i.e., they have no common factors. Then

$$2 = \frac{a^2}{b^2} \quad \text{and} \quad 2b^2 = a^2,$$

and we see that a^2 is even. But then we know from the above Lemma that a is also even, so $a = 2k$ with $k \in \mathbb{Z}$ and

$$a^2 = (2k)^2 = 4k^2 = 2b^2 \quad \text{or} \quad b^2 = 2k^2.$$

Since b^2 is even, it follows that b is also even. We have now shown that both a and b are even, but this contradicts the assumption that a and b are relatively prime. $\square$

Countable

- We say that $f\colon X \to Y$ is a bijection if it is one-to-one and onto. That means that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ and that $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.
- We say that two sets X and Y have the same cardinality if there is a bijection $f\colon X \to Y$.

Countable 1

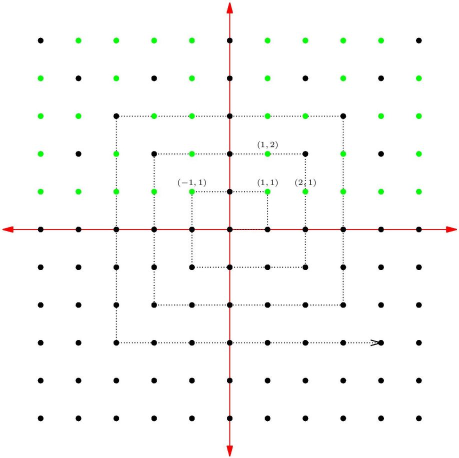
- We say that X is countably infinite if there is a bijection $f \colon \mathbb{N} \to X$.
- We say that X is countable if there is a surjection $f \colon \mathbb{N} \to X$. This means that we can write the elements of X as a list.
- A countable set is either countably infinite or finite.

Countable 2

- The set of integers, $\mathbb{Z}$ is countable, since
$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}.$$
- The set of rational numbers, $\mathbb{Q}$, is countable. This can be seen in many ways.

Countable 3

- The rational numbers a/b correspond to the pair (a, b), so $\mathbb{Q}$ corresponds to $\mathbb{Z} \times (\mathbb{Z} - \{0\})$.



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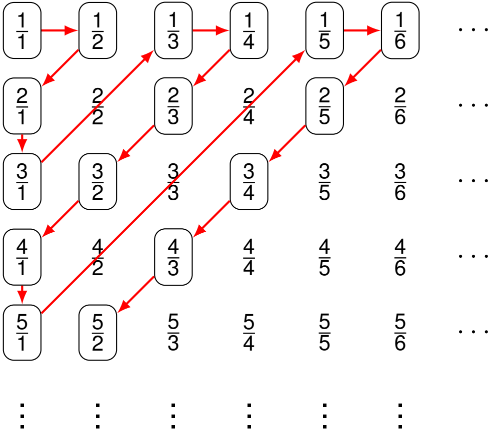
Countable 4

- 

- This shows that the set of positive rational numbers is countable. By alternating between positive and negative numbers, we can show that the whole of $\mathbb{Q}$ is countable.
- This picture also shows that a countable union of countable sets is countable.

Countable 5

- In 1874, Georg Cantor (1845 -– 1918) proved that $\mathbb{R}$ is not countable.
- Assume that $\mathbb{R}$ is countable. Then $[0,1]$ is also countable, and we can write $[0,1] = \{r_1, r_2, \ldots\}$ where $r_i = 0.d_{i1}d_{i2}\ldots$.

$$
\begin{array}{l}
r_1 = 0.\ d_{11}\ d_{12}\ d_{13}\ d_{14}\ d_{15}\ \cdots \\
r_2 = 0.\ d_{21}\ d_{22}\ d_{23}\ d_{24}\ d_{25}\ \cdots \\
r_3 = 0.\ d_{31}\ d_{32}\ d_{33}\ d_{34}\ d_{35}\ \cdots \\
r_4 = 0.\ d_{41}\ d_{42}\ d_{43}\ d_{44}\ d_{45}\ \cdots \\
r_5 = 0.\ d_{51}\ d_{52}\ d_{53}\ d_{54}\ d_{55}\ \cdots \\
\quad \vdots
\end{array}
$$

- $r = 0.\ d_1\ d_2\ d_3\ d_4\ d_5\ \cdots$
- We then construct a number $r = 0.d_1d_2\ldots$, where $d_i \neq d_{ii}$ and $d_i \neq 9$. Then $r \neq r_i$ for all i, and r is not in the list.