Points of Inflection, Three-point Secants, and Terrace Points

Helmer Aslaksen
University of Oslo
0317 Oslo, Norway
helmer.aslaksen@gmail.com

Christoph Kirfel
University of Bergen
5020 Bergen, Norway
Christoph.Kirfel@uib.no

Abstract

There are several ways to define a point of inflection. We will study four possible definitions and show that in general they are not equivalent. We prove a chain of implications between them, and construct counterexamples to all the converse implications. However, we show that for a certain class of functions they are all equivalent. Our definition of point of inflection requires a tangent, since that is necessary for constructing our chain of implications. Three of the possible definitions have been studied carefully in the past, but with slightly different assumptions. However, our concept of three-point secants seems to be new. We also discuss whether a function with a crossing zero must be locally monotonic around the zero and whether a point of inflection with horizontal tangent must be a terrace point. Finally we address some misconceptions among calculus students and give counterexamples to these misconceptions. However, we show that if we restrict ourselves to the above class of functions, the misconceptions are actually true.

1 Introduction

There are several ways to define a point of inflection. The usual definition is to say that f'' changes sign, but some authors instead require an extremum for f' or a crossing tangent. In this article, we also consider what we call the three-point secants property, which could also be used as a definition, and show that these four possible definitions of point of inflection are in general not equivalent. However, we show that if we restrict ourselves to functions that are twice differentiable in a punctured neighborhood around a point c, with f'' changing sign at c, and having a (possibly vertical) tangent at c, then we have a chain of implications between these properties, and if we consider the class of functions where f'' is continuous and only has isolated zeros, then these four properties are in fact equivalent. Two of the three alternative definitions require f to have a tangent at c, and for the third one, we need a tangent to show that it holds at a point of inflection. This illustrates why we believe it is natural to require that there should be a tangent at a point of inflection, which some authors do not require.

We also show that a function that is locally monotonic around a zero must have a crossing zero, and that a point of inflection with a horizontal tangent is a terrace point, but that the converse of both statements only holds for the above class of functions.

Finally we address some misconceptions among calculus students and give counterexamples to these misconceptions. However, we show that if we restrict ourselves to the above class of functions, the misconceptions are actually true.

This article extends the results of [2, 5, 7, 9, 11], who studied three of the properties. However, we also consider a fourth property and terrace points, and we feel that our focus on the existence of a tangent in the definition gives new insight.

2 Preliminaries

We first make some preliminary definitions.

Definition 1. Let f be defined on an open interval (a,b), except possibly at a point $c \in (a,b)$. We say that f **changes sign at** $\boldsymbol{c}$ if we can find points a' and b' with $a \le a' < c < b' \le b$ such that f has opposite signs on (a',c) and (c,b'). We say that f **has the same sign around** $\boldsymbol{c}$ if we can find a', b' as above such that f has the same sign on (a',c) and (c,b').

In most of our examples, f is defined at c, but we sometimes need to consider the more general situation.

Definition 2. Let f be defined on an open interval (a,b) with $c \in (a,b)$, and suppose that $f(c) = 0$. We call c a **crossing zero** if f changes sign at c, and a **kissing zero** if f has the same sign around c. We say that f has an **isolated zero** if we can find a', b' as above such that f does not have any other zeros than c on (a',b').

Notice that if f is continuous with $f(c) = 0$, then c is either a crossing, a kissing or a non-isolated zero. Notice also that changing sign at c is not the opposite of having the same sign around c. To see that, we will introduce an important family of functions.

$$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

These functions have been studied by [5] and [6] among others. We see that f_1 does not change sign at 0, but neither does it have the same sign around 0 (Figure 1). Since it oscillates between positive and negative values on both sides of $c = 0$, it changes sign infinitely often near c. This linguistic paradox of a function not changing sign *at c* because it changes sign infinitely often *near c* is a crucial mathematical point.

Since $\lim_{x\to 0} \frac{f_1(x)-f_1(0)}{x-0} = \lim_{x\to 0} \sin(1/x)$ does not exist, f_1 is not differentiable at 0, but since $\lim_{x\to 0} \frac{f_2(x)-f_2(0)}{x-0} = \lim_{x\to 0} x\sin(1/x) = 0$, f_2 is differentiable at 0. Notice

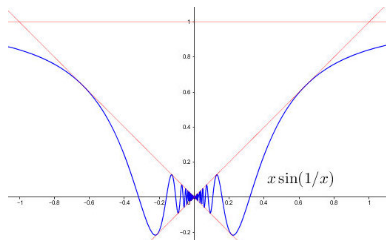

Figure 1: $f_1(x) = x\sin(1/x)$

that near the origin, the function oscillates between the two parabolas $y = \pm x^2$, while for large x, it approaches the skew asymptote $y = x$ (Figure 2). We have

$$f_2'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

but $\lim_{x\to 0} f_2'(x)$ does not exist, so $f_2(x)$ is not continuously differentiable at 0.

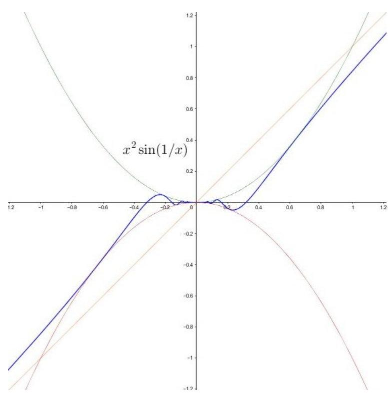

Figure 2: $f_2(x) = x^2\sin(1/x)$

For the rest of this paper, we write this kind of functions as just $x^n\sin(1/x)$, and assume that the value is 0 for $x = 0$. These functions will be useful for constructing counterexamples. Our guiding principle for constructing counterexamples is to start with a function that has the property we want, and then add an $x^n\sin(1/x)$ term to create an oscillating graph that breaks the property. We can for example now easily construct a differentiable function with an extremum where the derivative does not change sign. We start with the simplest example of a strict minimum, namely a parabola, and add an oscillating term of the form $x^2\sin(1/x)$. However,

$$f(x) = x^2 + x^2\sin(1/x) = x^2\Big(1 + \sin(1/x)\Big),$$

has infinitely many zeros that are all global extrema. We therefore "lift" the graph off the x-axis by reducing the amplitude of the oscillating term, and we choose for instance

(see Figure 3)

$$h_1(x) = x^2 + 1/2\,x^2 \sin(1/x) = x^2\Big(1 + 1/2 \sin(1/x)\Big).$$

Since

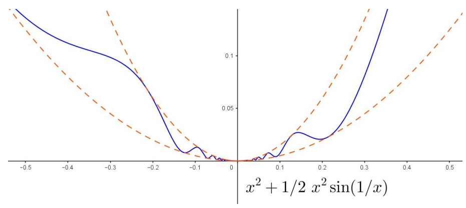

Figure 3: $h_1(x) = x^2 + 1/2\,x^2 \sin(1/x)$

$$h_1'(x) = 2x + x\sin(1/x) - 1/2\cos(1/x),$$

we see that h_1' oscillates between positive and negative values near $\pm 1/2$ arbitrarily close to 0, so that h_1' does not change sign *at* 0, since it changes sign infinitely often *near* 0. You may think that the problem is that h_1 is not twice differentiable, but we now give another counterexample that is actually twice continuously differentiable. In fact, in Theorem 12, we show that the converse implication is true if f'' is continuous and has only isolated zeros, so it is the nature of the zero of the second derivative that is crucial. We will therefore in the rest of this article always give two counterexamples if our first is not twice continuously differentiable.

For our twice continuously differentiable counterexample, we need at least x^5 in front of the sine term, and we might want to try

$$f(x) = x^2 + x^6 \sin(1/x), \tag{1}$$

which gives

$$f'(x) = x\Big(2 + 6x^4 \sin(1/x) - x^3 \cos(1/x)\Big).$$

However, here the derivative changes sign, since the trigonometric terms are all dampened by monomial terms. We therefore instead try

$$k_1(x) = x^6 + 1/2\,x^6 \sin(1/x), \tag{2}$$

which gives

$$k_1'(x) = x^4\Big(6x + 3x\sin(1/x) - 1/2\cos(1/x)\Big),$$

and we see that k_1 has a minimum at 0, but that k_1' does not change sign around 0, since the term in the parenthesis oscillates. Notice that in (1) we did not include the $1/2$ coefficient, since the power was higher in front of the sine term than in the monomial. It is only when they are equal, that we need to lift the sine term off the x-axis. We have thus proved the following theorem, which was also discussed in [3].

Theorem 1. *If f is differentiable on (a,b) and f' changes sign at $c \in (a,b)$, the c is a local extremum. The function*

$$h_1(x) = x^2 + 1/2\, x^2 \sin(1/x)$$

shows that the converse is not true.We call h_1 an ***oscillating parabola****. The function*

$$k_1(x) = x^6 + 1/2\, x^6 \sin(1/x)$$

is a counterexample that is also twice continuously differentiable.

3 Points of Inflection

We need one more concept before we can define a point of inflection.

Definition 3. Let f be continuous on an open interval (a,b) with $c \in (a,b)$. We say that f has a **vertical tangent** at c if $\lim_{h\to 0}\big(f(c+h) - f(c)\big)/h = \infty$ or $-\infty$. We say that f **has a tangent** at c if either f is differentiable at c or f has a vertical tangent at c.

There is no standard definition of point of inflection ([4, 5]), but we will use the following.

Definition 4. Let f be continuous on an open interval (a,b) with $c \in (a,b)$ and assume that f is twice differentiable except possibly at c, but that f has a tangent at c. We call c a **point of inflection** if f'' changes sign at c.

Let us start by discussing the main alternative options. Some authors define it in terms of a change of strict convexity at c, without explicitly considering the second derivative. They define convexity by saying that the graph lies below the secant line, or by saying that f' is monotonic ([12]). It follows from a theorem of Aleksandrov ([1]) that a convex function is twice differentiable almost everywhere, and it is not easy to construct an example of a strictly convex functions that fails to be twice differentiable at infinitely many points, and students are not likely to come across such functions, so we do not see any significant loss in generality by requiring f to be twice differentiable except possibly at c. This allows us to compare and contrast extrema and points of inflection in terms of f' and f'', respectively.

Some authors ([8]) simply require a change in non-strict convexity, which gives a straight line infinitely many points of inflection, just like a constant function has infinitely many extrema. We feel, however, that these two cases are different. An extremum is simply defined by the inequality $f(x) \geq$ (or $\leq$) $f(c)$, while a point of inflection is meant to indicate an actual inflection in the graph.

Some authors ([13]) define a point of inflection to be a point where the tangent crosses the graph. We will see later that this is not equivalent to our definition.

Some authors define a point of inflection to be a point where the tangent is steepest or least steep, that is f' has an extremum at c. This is discussed in [7], and we will see later that this is not equivalent to our definition.

These two alternative properties have been discussed carefully by several authors ([2, 5, 7, 9, 11]). In this article we will also consider another property, which we call tree-point secants. There is also the question of what conditions you want to put on the function at the point c. We have not found anybody who does not require f to be both defined and continuous at c, but for instance [12] does not require f to be differentiable at c, while for instance [2] requires f to be differentiable at c. In this article we take a middle position and require f to have a tangent at c. Since convexity is about how the graph lies with respect to the tangent, and a point of inflection is about change in convexity from left to right, it makes sense philosophically to assume that the tangents on the left and right agree at c. In addition, we will show that if f has a tangent at c, then a point of inflection will imply that f also satisfies these three other properties, and in Theorem 12, we will show that for functions whose second derivatives are continuous with only isolated zeros, the four properties are in fact equivalent. We feel that this is an important point, and we will therefore not say that

$$f(x) = \begin{cases} x^2 + x & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0 \end{cases}$$

has a point of inflection at 0, since the one-sided tangents do not coincide at 0 (see Figure 4).

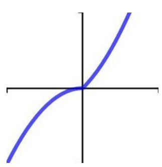

Figure 4: No tangent

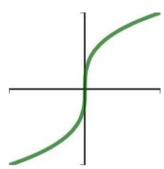

Figure 5: $f(x) = x^{1/3}$, vertical tangent

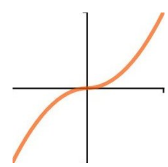

Figure 6: No second derivative

However, we say that $x^{1/3}$ has a point of inflection at 0, even though $f'(0)$ does not exist, because f has a vertical tangent at 0 (see Figure 5).

The following example shows a point of inflection at 0, even though $f''(0)$ does not exist. The left and right first derivatives match, so that $f'(0)$ exists, but the left and right second derivatives do not match (see Figure 6)

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

However, if f'' does exist at a point of inflection, c, then we can use the Mean Value Theorem to show that $f''(c)$ must equal 0. We have

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} f''\big(t(h)\big),$$

where $t(h)$ lies between c and $c + h$. Since f'' changes sign at c, one of the one-sided limits is greater than or equal to zero and the other is less than or equal to zero, so if the second derivative exists, it follows that $f''(c) = 0$.

Notice that if $f''(c) = 0$ and c is a point of inflection, then c is automatically an isolated zero of f'', since we require that f'' changes sign at c. This observation will be important later on, and explains why functions of the form $x^n \sin(1/x)$ do not have a point of inflection at 0.

We now define a series of properties.

Definition 5. Let f be twice differentiable on a punctured interval $(a,b)-\{c\}$ and with a tangent at c. We consider the following properties.

- **W1 (f'' changes sign):** f'' changes sign at c.
- **W2 (f' extremum):** The first derivative has a strict local extremum at c or $\lim_{h\to 0}\big(f(c+h)-f(c)\big)/h = \infty$ or $-\infty$.
- **W3 (crossing tangent):** The tangent crosses the curve at c with an isolated crossing point.
- **W4 (three-point secants):** There is an interval I around c such that for all $x_1 \in I$ on one side of c, there is an $x_2 \in I$ on the other side of c, such that $\big(x_2, f(x_2)\big)$ lies on the secant through $\big(x_1, f(x_1)\big)$ and $\big(c, f(c)\big)$.
- **W5 ($f'' = 0$):** $f''(c) = 0$ if it exists.

We have defined c to be a point of inflection if f satisfies W1 (f'' changes sign), while W2 (f' extremum) has been studied in [7], and [13] used W3 (crossing tangent). The relationship between the first three properties have been studied by several authors ([2, 5, 7, 9, 11]), but we have not seen W4 (three-point secants) used elsewhere, even though it is somewhat related to one of the properties studied in [4].

In Theorems 3, 4, 5, and 6 we show that for functions that are twice differentiable on a punctured interval $(a,b)-\{c\}$ and with a tangent at c, we have a chain of implications among these properties. We know that W5 ($f'' = 0$) is not equivalent to the other properties, since x^4 is an elementary counterexample, and in Theorems 7, 8, and 10 we show that the converses of the other implications are also false. However, in Theorem 12, we show that for functions that are twice continuously differentiable on an interval (a,b) and with only isolated zeros of f'', the first four properties are equivalent.

It is possible to define a point of inflection without requiring a tangent, and only using W1 (f'' changes sign). However, by requiring a tangent line, we get the three additional properties W2 (f' extremum), W3 (crossing tangent), and W4 (three-point secants), and in Theorem 12, we will show that these properties are equivalent to a point of infection for an interesting class of functions. In our view, this is a strong argument for requiring that there should be a tangent at a point of inflection.

We know that W1 (f'' changes sign) implies that f' has an extreme point. However, points of inflection and extreme points are fundamentally different in a subtle way. The actual definition of strict local extremum is that the value of f is extreme, and f' changing sign is a necessary, but not sufficient conditions. On the other hand, the actual definition of point of inflection is W1 (f'' changes sign), and we show that W2 (f' extremum), W3 (crossing tangent), W4 (three-point secants), and W5 are necessary, but not sufficient conditions.

For people living in countries with ski-jumps, the landing part of the hill is a great example of a point of inflection (see Figure 7). Unfortunately, in the sport it is called the critical point, which does not agree with the mathematical definition of a critical point.

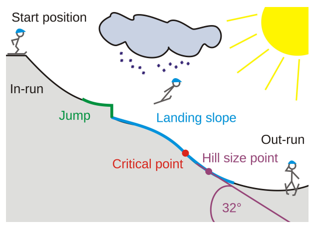

Figure 7: Ski jumping hill ([15])

We now want to state a series of theorems about implications among the W properties, but we first need to recall an interesting theorem, which unfortunately is often not covered in basic courses in analysis, namely Darboux's Theorem ([9]). We will therefore give a quick proof of it.

Theorem 2. *If $f(x)$ differentiable on the interval $[a,b]$ and $f'(a) < k < f'(b)$ (or $f'(a) > k > f'(b)$), then there is a point x_0 with $a < x_0 < b$ and $f'(x_0) = k$.*

Proof. By considering $f(x)-kx$, we can assume that $k = 0$. Since $f'(a) < 0$, there is a δ such that $a < x < a+\delta$ implies that $\big(f(x)-f(a)\big)/(x-a) < 0$, and therefore $f(x) < f(a)$. Similarly, there is a δ' such that $b-\delta' < x < b$ implies that $\big(f(b)-f(x)\big)/(b-x) < 0$, and therefore $f(x) < f(b)$. Since f is continuous, it must have a minimum on $[a,b]$, and it now follows that it must be at an interior point, and the derivative must be 0 there. □

We can now state the following useful corollary.

Corollary 1. *If f is differentiable on (a,b), then f' cannot have a removable or a jump discontinuity.*

Proof. If $L = \lim_{x\to c} f'(x) \neq f'(c)$, then we can without loss of generality assume that $f'(c) > L$, set $\epsilon = (f'(c)-L)/2$, and find a δ such that if $0 < |x-c| < \delta$, then $|f'(x)-L| < \epsilon$. If we choose $t \in (c, c+\delta)$, then $f'(t) < L+\epsilon < f'(c)$, but there is no point in (c,t) where $f'(x)$ equals $L+\epsilon$, which contradicts Darboux's Theorem. We can conclude that f' cannot have a removable discontinuity.

If $A = \lim_{x\to c^-} f'(x) \neq \lim_{x\to c^+} f'(x) = B$, then we can without loss of generality assume that $A < B$, set $\epsilon = (B-A)/2$, and find a δ such that if $0 < c-x < \delta$, then $|f'(x)-A| < \epsilon$, and if $0 < x-c < \delta$, then $|f'(x)-B| < \epsilon$. If we choose $t_1 \in (c-\delta, c)$ and $t_2 \in (c, c+\delta)$, then $f'(t_1) < A+\epsilon = B-\epsilon < f'(t_2)$. However, there is no point in (t_1, t_2) where $f'(x)$ equals $A+\epsilon = B-\epsilon$, which contradicts Darboux's Theorem. We can conclude that f' cannot have a jump discontinuity. □

We are now ready to state the first step in our chain of implications.

Theorem 3. *Let f be twice differentiable on a punctured interval $(a,b)-\{c\}$ and with a tangent at c. If f satisfies W1 (f'' changes sign), then it satisfies W2 (f' extremum).*

Proof. If f' is continuous at c, then this follows easily from the Mean Value Theorem, but how do we know that f' is continuous? The proof consists of showing that either f' is continuous at c, or there is a vertical tangent at c. We know that f' is increasing on one side of c and decreasing on the other, and therefore both $\lim_{x\to c^+} f'(x)$ and $\lim_{x\to c^-} f'(x)$ exist or are equal to $\pm\infty$. It then follows from Corollary 1 that either both one-sided limits exist and therefore f' is continuous at c, or there is a vertical tangent at c. ◻

Theorem 4. *Let f be twice differentiable on a punctured interval $(a,b)-\{c\}$ and with a tangent at c. If f satisfies W2 (f' extremum), then it satisfies W3 (crossing tangent).*

Proof. We can assume without loss of generality that f' has a strict minimum at c. If x is to the right of c, we use the Mean Value Theorem to show that $f(x)-f(c) = f'(d)(x-c)$ where d is between c and x. Since f' has a strict minimum at c, we have $f'(d) > f'(c)$, and it follows that

$$f(x) = f(c) + f'(d)(x-c) > f(c) + f'(c)(x-c),$$

which shows that f lies above the tangent to the right. A similar argument works for x to the left of c, and we can conclude that we have a crossing tangent. ◻

To show that W3 (crossing tangent) implies W4 (three-point secants) is a bit more complicated, and we first prove a lemma. The idea behind Lemma 1 is that the graph of a function cannot be separated from the tangent by a secant (see Figure 8).

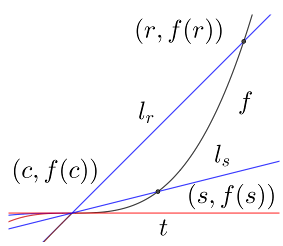

Figure 8: Secant lemma

Lemma 1. *Assume that f is continuous on (a,b) with a tangent at $c \in (a,b)$, and that there is $r \in (c,b)$ such that $\big(r, f(r)\big)$ does not lie on the tangent. Consider the secant l_r from c to r. Then there exists a point $s \in (c,r)$ so that the secant l_s from c to s lies between the secant l_r and the tangent.*

Proof. We first assume that the tangent, t, is not vertical, and that the slope, p, of the secant l_r is larger than the slope, $f'(c)$, of the tangent. Assume that the lemma is false, i.e., there is no point s, so that l_s lies between l_r and t. Then for all $s \in (c,r)$, the point $\big(s, f(s)\big)$ must lie on or above the secant l_r. Since the equation of l_r is given by $y = f(c) + p(x-c)$, we get $f(s) \geq f(c) + p(s-c)$ and

$$\frac{f(s)-f(c)}{s-c} \geq \frac{f(c)+p(s-c)-f(c)}{s-c} = p > f'(c),$$

and if we take the limit when s approaches c, we would get the contradiction $f'(c) > f'(c)$. When the slope of the secant is less than the slope of the tangent, the inequalities are simply reversed. A similar argument also holds if the tangent is vertical. In that case we must show that the limit of the fraction equals ∞ or $-\infty$. But that is impossible if the curve is separated from the tangent by a secant. $\square$

We now prove the next step in our sequence of implications among the W properties.

Theorem 5. *Let f be twice differentiable on a punctured interval $(a,b) - \{c\}$ and with a tangent at c. If f satisfies W3 (crossing tangent), then it satisfies W4 (three-point secants).*

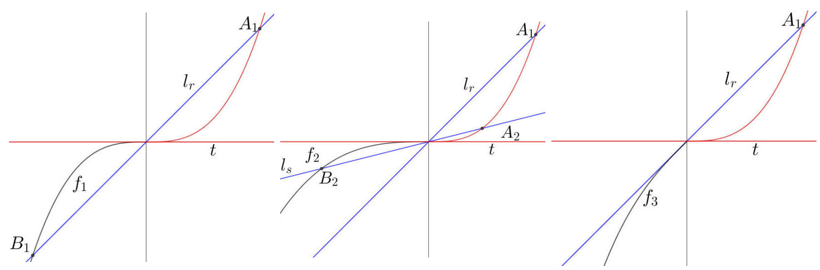

Figure 9: Three-point secants

Proof. Let t be the tangent at c, and assume that there exists a neighborhood (a,b) such that f lies below the tangent for $x \in (a,c)$ and above for $x \in (c,b)$. We now select $r \in (c,b)$ on the right and consider the secant l_r through $A_1 = \big(r, f(r)\big)$ and $\big(c, f(c)\big)$. In order for f to satisfy the three-point secants property, we must show that l_r also intersects the graph of f on the left. For this purpose, we now consider three possibilities f_1, f_2 and f_3 for the graph of f on the left (see Figure 9). In Figure 9, the tangent is the x-axis, and the point of inflection is at the origin. Either there is an $s \in (a,c)$ so that $B_1 = \big(s, f(s)\big)$ lies on l_r (see the graph f_1), or the graph of f for $x \in (a,c)$ lies completely between the tangent t and the secant l_r (see the graph f_2) or the graph of f for $x \in (a,c)$ lies below the secant l_r (see the graph f_3). However, it follows from Lemma 1 that the f_3 case is impossible, since then the secant would separate the graph from the tangent. In the f_2 case, we can pick any point $s \in (a,c)$ and consider the secant l_s from $B_2 = \big(s, f(s)\big)$ to $\big(c, f(c)\big)$. Since l_s lies between l_r and t on the left, it lies between t and l_r on the right, too. There are now three possible cases. Either the graph is always below l_s on the right, always above l_s on the right, or it intersects l_s on the right. The graph cannot lie below, since it intersects l_r in A_1, and it cannot lie above, because then it would be separated from the tangent by a secant, which is impossible by Lemma 1. We can therefore find $r' \in (c,b)$ such that $A_2 = \big(r', f(r')\big)$ lies on l_s. So in either case we now have a pair, which we simply denote as r and s, such that $\big(r, f(r)\big)$, $\big(c, f(c)\big)$ and $\big(s, f(s)\big)$ lie on a common secant l.

If f is differentiable at c, then the slope of the tangent is the limit of the slopes of nearby secants, so we can find a neighborhood (a',b') with $s \le a' < c < b' \le r$, such that for $x \in (a',b')$, the secant l_x from $\big(x, f(x)\big)$ to $\big(c, f(c)\big)$ lies between t and l. But in that case it is clear that l_x intersects the graph of f on the other side of c, too, so W4 (three-point secants) holds.

If f is not differentiable at c, but has a vertical tangent at c, then the tangent is still the limit of the slope of nearby secants, so the same argument as above holds. □

Notice that if we had not included the existence of a tangent at c as part of the definition of a point of inflection, we would not have been able to conclude that there are three-point secants around a point of inflection (see Figure 4). The fact that W2 (f' extremum) and W3 (crossing tangent) both require f to have a tangent at c, and that Theorem 5 shows that W4 (three-point secants) holds if f has a tangent at c, indicates that it is natural to require that f should have a tangent at a point of inflection.

We can now prove the last step in our chain of implications.

Theorem 6. *Let f be twice differentiable on a punctured interval $(a,b) - \{c\}$ and with a tangent at c. If f satisfies W4 (three-point secants), then it satisfies W5 ($f'' = 0$).*

Proof. The three-point secants property shows that for all $\epsilon > 0$, there exists points a, b with $c - \epsilon < a < c < b < c + \epsilon$ such that $\big(a, f(a)\big)$, $\big(c, f(c)\big)$ and $\big(b, f(b)\big)$ lie on the secant. The Mean Value Theorem then gives us points a' and b' between a and c, and b and c, respectively such that

$$f'(a') = \frac{f(c) - f(a)}{c - a} = \frac{f(b) - f(c)}{b - c} = f'(b').$$

This then implies that

$$f''(c) = \lim_{b' \to c^+} \frac{f'(b') - f'(c)}{b' - c} = \lim_{a' \to c^-} \frac{f'(a') - f'(c)}{a' - c}.$$

Since $f'(a') = f'(b')$, the numerators are equal, while the denominators have opposite sign, so the limits must have opposite sign, and we get $f''(c) = 0$. □

We will now see that the converses of Theorems 3, 4, 5, and 6 are false. As explained previously, we will in each case make sure that we also have a counterexample that is twice continuously differentiable.

First of all, $f(x) = x^4$ satisfies W5 ($f'' = 0$), but none of the other properties, so the converse of Theorem 6 fails.

Theorem 7. *The function $f_2(x) = x^2 \sin(1/x)$ shows that W4 (three-point secants) does not imply W3 (crossing tangent). The function $f_4(x) = x^4 \sin(1/x)$ is a counterexample that is also twice continuously differentiable.*

Proof. Both f_2 and f_4 have the x-axis as the tangent at $x = 0$, and since they are odd functions, they automatically satisfy W4 (three-point secants). However, since they take on both positive and negative values near the origin, they do not satisfy W3 (crossing tangent), see Figure 2. □

We now want to construct a counterexample to see that W3 (crossing tangent) does not imply W2 (f' extremum), and we follow the principle used when discussing the oscillating parabola, and start with a function with the stated property, namely that f has a crossing tangent, and add an oscillating term. The simplest example of a crossing tangent is x^3, so we examine functions of the form $f(x) = x^3 + ax^n \sin(1/x)$. If $n < 3$, then the oscillations are so big that we will not get a crossing tangent, and if we set $n = 3$ and $a = 1$, then the curve touches the tangent infinitely many times. We therefore set (see Figure 10)

$$h_2(x) = x^3 + 1/2\,x^3 \sin(1/x) = x^3\Big(1 + 1/2 \sin(1/x)\Big).$$

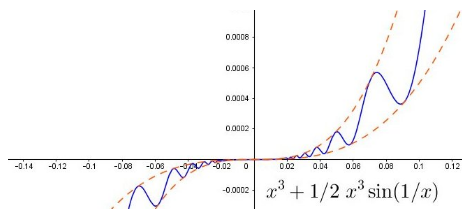

Figure 10: $h_2(x) = x^3 + 1/2\,x^3 \sin(1/x)$

Then

$$h_2'(x) = 1/2\,x\Big(6x + 3x \sin(1/x) - \cos(1/x)\Big), \tag{3}$$

and since the parenthesis consists of terms involving x and an undamped oscillating term, we see that the parenthesis changes sign infinitely often near 0. Hence h_2' takes on both positive and negative values near 0, and 0 is not an extremum of h_2'.

To construct a counterexample that is twice continuously differentiable, we will need at least x^5 in front of the sine term, and as we saw when discussing k_1 in Equation (2), we need the same power in the monomial, too, so we set

$$k_2(x) = x^5 + 1/2\,x^5 \sin(1/x),$$

which gives

$$k_2'(x) = 1/2\,x^3\Big(10x + 5x \sin(1/x) - \cos(1/x)\Big).$$

Again we see that k_2 has a crossing tangent at 0, but that k_2' does not have an extremum at 0. We have thus proved the following theorem.

Theorem 8. *The function*

$$h_2(x) = x^3 + 1/2\,x^3 \sin(1/x)$$

shows that W3 (crossing tangent) does not imply W2 (f' extremum). We call h_2 an ***oscillating cubic****. The function*

$$k_2(x) = x^5 + 1/2\,x^5 \sin(1/x)$$

is a counterexample that is also twice continuously differentiable.

Notice that there is a subtle difference between the oscillating parabola and the oscillating cubic. The oscillating parabola has an extremum, because extremum is defined geometrically and not analytically, i.e., in terms of an inequality and not in terms of f' changing sign. The oscillating cubic, on the other hand, does not have a point of inflection, since that is defined analytically, and not in terms of geometric properties like W3 (crossing tangent).

The oscillating cubic illustrates a common misconception, which we summarize in the following theorem.

Theorem 9. *Let f be differentiable on (a,b) with a zero at $c \in (a,b)$. If f is locally strictly monotonic around c, then c is a crossing zero. However, the converse is not true, and a counterexample is given by the oscillating cubic (see Figure 10)*

$$h_2(x) = x^3 + 1/2\,x^3 \sin(1/x).$$

Proof. The first part is immediate, and h_2 obviously has a crossing zero, but the derivative (3) changes sign all the time around 0, so h_2 is not monotonic on any neighborhood around 0. □

We will now continue finding counterexamples to the converse implications among the W properties. To show that W2 (f' extremum) does not imply W1 (f'' changes sign), we could simply integrate $h_1(x) = x^2 + 1/2\,x^2 \sin(1/x)$, which was an example of a function with an extremum where the derivative did not change sign. However, it is not easy to find a simple anti-derivative of h_1. We would also like to relate the discussion to the concept of terrace point. This is not a standard term in calculus books, but has recently been popularized by for instance ([10]).

Definition 6. Let f be differentiable on an open interval (a,b) with $c \in (a,b)$. We call c a **terrace point** if $f'(c) = 0$, and f' has the same sign around c.

We will first prove a simple lemma. Recall that $h_2(x) = x^3 + 1/2\,x^3 \sin(1/x)$ was not increasing. However, if we add a monomial x^m with $m \geq 1$ in front of the sine term, the function becomes increasing.

Lemma 2. *If $n \geq 3$ is odd and $m \geq 1$, then*

$$f(x) = x^n\Big(1 + x^m \sin(1/x)\Big),$$

is increasing for all x, and 0 is a terrace point. If $m \geq 3$, then 0 is a point of inflection.

Proof. We have

$$f'(x) = x^{n-1}\Big(n + (n+m)x^m \sin(1/x) - x^{m-1}\cos(1/x)\Big),$$

and since $n-1$ is even, we simply need to show that the parenthesis is positive in order to conclude that f is increasing. If $m-1 \geq 1$, the parenthesis will be close to n when x is small, and if $m = 1$, we need to show that

$$\Big(n + (n+1)x\sin(1/x) - \cos(1/x)\Big) > 0.$$

If $|x| < 1/\pi$, then $|x\sin(1/x)| < |x| < 1/\pi$ and if $|x| \geq 1/\pi$, then $|1/x| \leq \pi$. Since $1/x$ is the argument of the sine function, we see that x and $\sin(1/x)$ have the same sign and the product is nonnegative, so $0 \leq x\sin(1/x)$ for $|1/x| \leq \pi$ (see Figure 1). Combining this, we see that $-1/\pi < x\sin(1/x)$ for all x, and hence

$$\begin{aligned} n + (n+1)x\sin(1/x) - \cos(1/x) &> n - (n+1)/\pi - 1 \\ &> n - (n+1)/3 - 1 = 2n/3 - 4/3 \geq 2/3, \end{aligned}$$

since $\pi > 3$ and $n \geq 3$, so f is increasing and 0 is a terrace point.

We have

$$\begin{aligned} f''(x) = x^{n-2}\Big(&n(n-1) + (n+m-1)(n+m)x^m\sin(1/x) \\ &- 2(m+n-1)x^{m-1}\cos(1/x) - x^{m-2}\sin(1/x)\Big), \end{aligned}$$

and since $n-2$ is odd, we simply need to show that the parenthesis is positive in order to conclude that f has a point of inflection. If $m-2 \geq 1$, then there are powers of x in front of all the trigonometric terms, so the parenthesis will be close to $n(n-1)$ when x is small, so f'' will change sign at 0. However, if $m-2 \leq 0$, we will not be able to dampen the last sine term with monomials, and f'' will oscillate. □

This shows that when the power in front of the sine term is higher than the power of the monomial, the amplitudes of the oscillations are so small that the function is still increasing, but if the power in front of the sine term is only one or two higher than the power of the monomial, there will still be enough oscillation to prevent a change in the sign of f'' at 0. However, if $m \geq 3$, the oscillations are so small that we get a point of inflection at 0. We will also not need to reduce the amplitude of the oscillating term by a factor of $1/2$, like we did for $h_2(x)$, and we define (see Figure 11)

$$h_3(x) = x^3 + x^4\sin(1/x) = x^3\Big(1 + x\sin(1/x)\Big).$$

Notice that in Figure 11 we have used different scales along the two axes to highlight

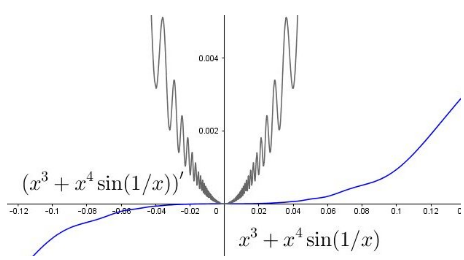

Figure 11: $h_3(x) = x^3 + x^4\sin(1/x)$

the oscillations in h_3'. We see that h_3' has a minimum and hence h_3 satisfies W2 (f' extremum). However, since

$$h_3''(x) = 6x + 12x^2\sin(1/x) - 6x\cos(1/x) - \sin(1/x), \tag{4}$$

and since the terms involving x are small and $\sin(1/x)$ oscillates, we see that h_3'' does not change sign at 0 since it changes sign infinitely often near 0. This counterexample was also given in ([Ko]), where it was also claimed that

$$f(x) = x^3 + x^6 \sin(1/x)$$

is a twice continuously differentiable counterexample. However, Lemma 2 shows that f'' does change sign around 0. This is probably just a typo in [Ko], since if we set

$$k_3(x) = x^5 + x^6 \sin(1/x),$$

then we can use Lemma 2 with $n = m = 3$ to conclude that k_3' has a minimum 0, but that k_3'' does not change sign at 0. We have thus proved the following theorem.

Theorem 10. *The function*

$$h_3(x) = x^3 + x^4 \sin(1/x)$$

shows that W2 (f' extremum) does not imply W1 (f'' changes sign). We call h_3 an ***asymptotic oscillating cubic****, since* $\lim_{x\to\infty}(h_3(x)/2x^3) = 1$. *The function*

$$k_3(x) = x^5 + x^6 \sin(1/x)$$

is a counterexample that is also twice continuously differentiable.

It turns out that h_3 (see Figure 11) is also a counterexample to another interesting result, so we would also like to state the following theorem.

Theorem 11. *A point of inflection with a horizontal tangent is a terrace point, but the asymptotic oscillating cubic $h_3(x) = x^3 + x^4 \sin(1/x)$ has a terrace point that is not a point of inflection at 0.*

Proof. If c is a point of inflection with $f'' > 0$ on an interval (c, b), then for $x \in (c, b)$, we can use the Mean Value Theorem to conclude that $f'(x) - f'(c) = f''(y)(x - c)$ for some point $y \in (0, x)$. Since $f'(c) = 0$ and $f''(y) > 0$, it follows that $f'(x) > 0$ for $x \in (c, b)$, and a similar argument shows that there is an interval (a, c) such that $f'(x) > 0$ for $c \in (a, c)$. The counterexample follows from Lemma 2. □

4 Functions with a continuous second derivative that has only isolated zeros

We observed at the beginning of this paper that for a continuous function, a zero is either a crossing zero, a kissing zero or a non-isolated zero. Except for x^4, all our counterexamples have had non-isolated zeros of f''. However, if f'' is continuous and c is a point of inflection with $f''(c) = 0$, then f'' changes sign, and c must therefore be an isolated zero of f''. In this section, we will therefore restrict ourselves to the following class of functions, which was also studied in ([7]),

$$V = \{f \mid f'' \text{ is continuous and has only isolated zeros}\}.$$

The key point is that for $f \in V$, a zero of f'' must either be a crossing zero or a kissing zero. However, notice that V does not exclude all oscillating functions of the form we have considered. It follows from Lemma 2 that functions of the form

$$f(x) = x^n(1 + x^m \sin(1/x)) \quad \text{with } n \text{ odd and } m \geq 3$$

are in V. The point is that in this case the oscillations are sufficiently small to not create infinitely many zeros of f''. Notice also that we exclude linear functions from V.

We can now prove a useful lemma, which shows that for our class V of functions, f, f', and f'' all have fixed sign on both sides of c. This simplifies many arguments, and is one of the reasons why we want to consider this family of functions.

Lemma 3. *If f'' is continuous and has only isolated zeros, then both f' and f also have only isolated zeros, and we can find an interval (a', b') around c such that f, f', and f'' all are nonzero on $(a', b') - \{c\}$ and have fixed sign on both sides of c. Moreover, if*

$$f'(c) = f(c) = 0,$$

then there are only two possible cases for $x > c$:

$$R1 : f > 0, f' > 0, f'' > 0,$$
$$R2 : f < 0, f' < 0, f'' < 0,$$

and there are only two possible cases for $x < c$:

$$L1 : f > 0, f' < 0, f'' > 0,$$
$$L2 : f < 0, f' > 0, f'' < 0.$$

So all together, there are four possible cases, by combining one from the left and one from the right.

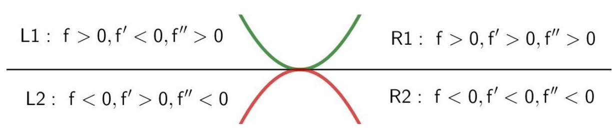

Figure 12: The four cases

Proof. Since f'' is continuous and has only isolated zeros, it must be nonzero on $(a', b') - \{c\}$ and have fixed sign on both sides of c. We now prove that f' also has only isolated zeros. Assume that f' has a non-isolated zero at $d \in (a', b')$. Then we can find $e \in (a', b')$ arbitrarily close to d with $f'(e) = 0$. By choosing e sufficiently close to d, we can assume that either $d = c$ or that d and e are on the same side of c. But then it follows from Rolle's Theorem that f'' must have a zero between d and e, and this zero cannot be at c, since c is not between d and e. But we know that f'' is nonzero on $(a', b') - \{c\}$, so it follows that f' has only isolated zeros. We then apply exactly the same argument to get that f also has only isolated zeros. Since f' and f'' are also both continuous and have only isolated zeros, it follows that they all are also both nonzero on $(a', b') - \{c\}$ and have fixed sign on both sides of c.

Notice that we cannot choose the signs of f, f', and f'' freely. We only get four cases, not eight. For instance, since $f(c) = 0$, we cannot have $f > 0$ and $f' < 0$ on the right, since the Mean Value Theorem would give us positive values of f' if f is positive. Similarly, since $f'(c) = 0$, we cannot have $f' > 0$ and $f'' < 0$ on the right. So we observe that on the right, the three functions must all have the same sign, while on the left, they must have alternating signs. □

Remark 1. The assumptions that $f'(c) = f(c) = 0$ are not as restrictive as it may seem, since by adding a linear function, we can make both $f(c)$ and $f'(c)$ equal to zero without affecting the value of $f''(c)$.

Remark 2. Notice also that the Rolle's Theorem argument only works one way. If f'' has only isolated zeros, then so does f', but the converse is not true, as can be seen from the asymptotic oscillating cubic h_3 (see Figure 11). The origin is a non-isolated zero for h_3'', but an isolated zero for both h_3' and h_3.

We now show that for functions in V, all the implications we have considered can be reversed, namely Theorems 5, 4, 3, 1, 9 and 11.

Theorem 12. *Let f'' be continuous with only isolated zeros on (a, b) and let $c \in (a, b)$. Then*

$$\begin{aligned}
\text{W4 (three-point secants)} &\Longrightarrow \text{W3 (crossing tangent)},\\
\text{W3 (crossing tangent)} &\Longrightarrow \text{W2 } (f' \text{ extremum}),\\
\text{W2 } (f' \text{ extremum}) &\Longrightarrow \text{W1 } (f'' \text{ changes sign}),\\
c \text{ is an extremum of } f &\Longrightarrow f' \text{ changes sign at } c,\\
c \text{ is a crossing zero} &\Longrightarrow f \text{ is locally strictly monotonic around } c, \text{ and}\\
c \text{ is a terrace point} &\Longrightarrow c \text{ is a point of inflection with horizontal tangent.}
\end{aligned}$$

Proof. Adding a linear function to f will not affect any of the conditions, so we can assume that $f'(c) = f(c) = 0$.

W4 (three-point secants) $\Longrightarrow$ W3 (crossing tangent). The result follows from Lemma 3, since we can only have L1&R2 or L2&R1.

W3 (crossing tangent) $\Longrightarrow$ W2 (f' extremum). We know that $f(x) > 0$ (< 0) for $x > c$ and $f(x) < 0$ (> 0) for $x < c$, and we can now use Lemma 3 to conclude that we must have case L2&R1 (L1&R2), and we see that f' has an extremum.

W2 (f' extremum) $\Longrightarrow$ W1 (f'' changes sign). If f' has an extremum, then we know that $f''(c) = 0$. If c is an extremum of f', then f' must be positive (negative) on both sides of c, and we see from Lemma 3 that f'' changes sign at c.

f extremum $\Longrightarrow$ f' changes sign. We can apply Lemma 3 to conclude that f' has fixed signs on both sides of c, which shows that f' changes sign at c.

If c is a terrace point, we can without changing the value of f'' add a constant to ensure that $f(c) = f'(c) = 0$, and therefore use Lemma 3. But then we see that if f' does not change sign at c, then f'' changes sign at c. □

With the exception of the three-point secants, similar results were also obtained in [2, 5, 7, 9, 11]. It is important to realize that the crucial aspect is not differentiability. This result does not hold for C^∞ functions, since we could construct infinitely differentiable counterexamples of the form

$$f(x) = \begin{cases} \left(e^{-x^{-2}} \sin(1/x)\right)^2 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\left(e^{-x^{-2}} \sin(1/x)\right)^2 & \text{if } x < 0. \end{cases}$$

If we instead consider analytic functions, i.e., functions that converge to their Taylor series, then they belong to V, except for linear functions, since their second derivatives are constantly equal to zero. However, as discussed earlier, we do not consider linear functions to have points of inflection, so that distinction does not affect our discussion.

5 Misconceptions in Calculus

Some misconceptions about points of inflections were studied in [14]. Calculus students are told about x^3 and x^4, and one of the goals of this article is to raise awareness of the more complex counterexamples discussed above. In this section, we therefore discuss some misconceptions related to these counterexamples. Our experience tells us that many calculus students have the following misconceptions, with counterexamples in parenthesis.

1. If f is differentiable at a strict, interior extremum, then f' changes sign, so if you want to find the extrema of a differentiable function on an open interval, you only need to find the points where f' changes sign. (Oscillating parabola $h_1 = x^2 + 1/2\,x^2 \sin(1/x)$, see Figure 3 and Theorem 1.)

2. If f has a crossing zero, then f must be locally monotonic. (Oscillating cubic $h_2 = x^3 + 1/2\,x^3 \sin(1/x)$, see Figure 10 and Theorem 9.)

3. A terrace point is a point of inflection. (Asymptotic oscillating cubic $h_3 = x^3 + x^4 \sin(1/x)$, see Figure 11 and Theorem 11.)

4. If $f'(c) = 0$, then c is either an extremum or terrace point. (Oscillating cubic $h_2 = x^3 + 1/2\,x^3 \sin(1/x)$, see Figure 10.)

5. If $f'(c) = f''(c) = 0$, then c is an extremum or a point of inflection. (Asymptotic oscillating cubic $h_3 = x^3 + x^4 \sin(1/x)$, see Figure 11.)

Why would a student hold these false beliefs? The opposite of all the claims are indeed true and if we assume that f'' is continuous and has only isolated zeros, then all the claims are indeed true! The first three misconceptions, which are related to Theorems 1, 9 and 11, all seem geometrically "obvious". Claim 1 is related to how students are taught to find extrema. Claim 2 makes sense verbally. The standard example of a terrace point is x^3, which indeed does have a point of inflection, and students may believe that this is a generic example, which would make Claim 3 true.

The last two claims are related to the first and second derivative test. They learn that if f' changes sign at c, i.e., c is a crossing zero of f', then f has either a maximum or a minimum, and they are shown x^3, which has a terrace point since f' has a kissing zero at c. They may then believe that every zero is either a crossing zero or a kissing zero, which would make Claim 4 true.

Claim 5 is about the case left open by the second derivative test. Strong students are familiar with x^4 and know that you do not necessarily get a point of inflection when $f'(c) = f''(c) = 0$. However, they may believe that x^4, x^3, and $-x^4$, i.e., minimum, point of inflection and maximum, are the only three possible cases, which would make Claim 5 true.

We therefore believe that it will be instructive for calculus students to learn about these counterexamples, and to learn under which conditions they can be avoided.

References

[1] A. D. Aleksandrov, *Almost everywhere existence of the second differential of a convex functions and related properties of convex surfaces,* Uchenye Zapisky Leningrad Gos. Univ. Math. Ser. **37** (1939), 3-–35 (in Russian).

[2] A. M. Bruckner, *Nonequivalent Definitions of Inflection Points,* Amer. Math. Monthly, **69** (1962), no. 8, 787–789, doi.org/10.2307/2310782.

[3] Rainer Danckwerts and Dankwart Vogel, *Analysis verständlich unterrichten,* Mathematik Primär- und Sekundarstufe, Springer Spektrum, 2006.

[4] G. M. Ewing, *On the Definition of Inflection Point,* Amer. Math. Monthly, **45** (1938), no. 10, 681–683, doi.org/10.2307/2302439.

[5] Jürgen Grahl and Shahar Nevo, *Oscillating Functions that Disprove Misconceptions on Real-Valued Functions,* Math. Mag., **92** (2019), 47–57, doi.org/10.1080/-0025570X.2019.1537415.

[6] H. Turgay Kaptanoglu, *In Praise of* $y = x^\alpha \sin(1/x)$*,* Amer. Math. Monthly, **108** (2001), no. 2, 144-–150, doi.org/10.2307/2695527.

[7] Duane Kouba, *Can We Use the First Derivative to Determine Inflection Points?,* College Math. J., **26** (1995), no. 1, 31–34, doi.org/10.1080/07468342.1995.-11973663.

[8] Tom Lindström, *Kalkulus* (1st ed.), Universitetsforlaget, 1994 (in Norwegian).

[9] Michael Olinick and Bruce B. Peterson, *Darboux's Theorem and Points of Inflection,* College Math. J., **7** (1976), no. 3, 5–9, doi.org/10.2307/3027142.

[10] Arnold Ostebee and Paul Zorn, *Calculus from graphical, numerical, and symbolic points of view : single variable* (2nd edition.), Houghton Mifflin Company, 2002.

[11] A. R. Rajwade and A. K. Bhandari, *Surprises and Counterexamples in Real Function Theory,* Texts and Readings in Mathematics, vol. 42, Hindustan Book Agency, 2007.

[12] Satunino L. Salas, Einar Hille and Garret J. Etgen, *Calculus: One and Several Variables* (eighth edition), Wiley, 1999.

[13] Michael Spivak, *Calculus* (third edition), Cambridge University Press, 1994.

[14] Pessia Tsamir and Regina Ovodenko, *University students' grasp of inflection points,* Educ. Stud. Math., **83** (2013), no. 3, 409–427, doi.org/10.1007/s10649-012-9463-1.

[15] `en.wikipedia.org/wiki/File:Ski_jumping_hill_schematic.svg`.