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Probability and Combinatorics in High School Mathematics

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My Background 1

- Bachelor degree from the Univ. of Oslo.
- PhD from the University of California, Berkeley.
- 22 years at the Department of Mathematics, National University of Singapore, 1989 – 2011.
- Introduced two new “General Education Modules” at the NUS: “Heavenly Mathematics and Cultural Astronomy” and “Mathematics in Art and Architecture”.
- Vice President of the Singapore Mathematical Society.
- Textbook Consultant for the Ministry of Education in Singapore.
- Chair of organizing committee for the Singapore Math Olympiad.

My Background 2

- Moved back to Norway in 2011 to take a joint position at the Department of Teacher Education and the Department of Mathematics at the University of Oslo.
- Introduced a new course at the University of Oslo, MAT4010 School Mathematics from an Advanced Standpoint.
- My interest lies between mathematics and mathematics education. I call it educational mathematics.

Why Talk about Probability and Combinatorics?

- Many teachers have a weak background.
- It is difficult to get partial credit. Either it is right or it is wrong.
- It is easy to make mistakes. Wrong arguments can look correct and correct arguments can look wrong!
- If you think you understand probability, then you don't understand it!
- That's why it's fun!

What Do I Want to Talk about Today? 1

- Sometimes we discuss things you can say to all the students.
- Sometimes we discuss things you can say to good students once in a while.
- Sometimes we discuss things you will probably never say to any students, but which give you an understanding that makes you feel confident when you give a simplified explanation to the students.

What Do I Want to Talk about Today? 2

- Sampling
- Paradoxes
- History of probability

What is the Difference Between Probability and Combinatorics?

- In combinatorics, the answer is always a whole number.
- In probability, the answer is a number between 0 and 1.
- You often use combinatorics to compute probabilities, but you must not confuse the concepts.
- I once moderated the draft of an exam paper in primary school teacher education, which asked about the number of ways you could get two aces when you draw five cards from a deck.
- They reassured us that we could assume that the deck was well shuffled!

Making a Test

- Many exercises, especially in lower secondary school, are confusing because it is (in my opinion) unclear whether the samples are ordered or unordered and with or without replacement.
- A Norwegian textbook asked the following question: “A teacher has 2 problems about algebra, 3 about geometry and 4 about probability, and wants to make a test with one problem of each type. How many different tests can the teacher make?”
- Discuss: Is it ordered or unordered?
- Remember Helmer’s Law: Anything that can be misunderstood by a student will at some stage be misunderstood.

Making a Test Discussion

- The problem with the test question is that from a practical point of view, there is a difference between a test where the first question is easy, and a test where the first question is hard. So the practical problem is best solved using an ordered model.
- However the authors of the book asked the question after introducing the combinatorial product rule, which assumes that the order does not matter.

Boy and Girl

- I have two children. I can either have two boys, two girls or one of each. (We ignore identical twins and other complications, and assume equal and independent birthrates.) I use the rule that says that the probability of an event E is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}},$$

  and get

$$P(\text{I have a boy and a girl}) = \frac{1}{3}.$$

- Discuss: Is this correct?

Boy and Girl Discussion

- The favorable/possible formula only applies to uniform distributions, where each outcome is equally likely. If we order the kids by age, the event "a boy and a girl" is the union of the two simple events "big brother and little sister" and "big sister and little brother".

big brother and little brother	big brother and little sister
big sister and little brother	big sister and little sister

- Using an ordered model, we get a uniform distribution, which allows us to use the favorable/possible rule, giving us the answer $2/4 = 1/2$.
- Notice that after having done the computations, we conveniently forget about the ordering.
- However, if we use the unordered model, we no longer get a uniform distribution. Two of the outcomes have probability 1/4 and one has probability 1/2. Hence we cannot use the favorable/possible formula.

Two Dice

a. How many outcomes can you get when you roll two regular dice?

b. Which of these outcomes is most likely to occur when you roll two dice?
   - $1, 6$,
   - $6, 6$,
   - They are equally likely.

- The answers are given as 36 and $1, 6$.
- Discuss: Can the questions be interpreted in different ways? Are there alternative answers? Are the answers consistent?

Two Dice Discussion 1

- In a) they mean ordered, in b) they mean unordered. Both interpretations are possible, but you need to be consistent.
- The question is whether you can differentiate between the dice.
- Assume that one die is yellow and the other is blue. Then there are 36 possible outcomes.

	Y-1	Y-2	Y-3	Y-4	Y-5	Y-6
B-1						
B-2						
B-3						
B-4						
B-5						
B-6						

Two Dice Discussion 2

- How many different outcomes are there if the dice are identical and are thrown at the same time in a dice cup?
- There are therefore $15 + 6 = 21$ outcomes. We have a non-uniform probability model, where 15 outcomes (those above the diagonal) have a probability of $1/18$ and 6 outcomes (those on the diagonal) have a probability of $1/36$.

	1	2	3	4	5	6
1	X	XX	XX	XX	XX	XX
2		X	XX	XX	XX	XX
3			X	XX	XX	XX
4				X	XX	XX
5					X	XX
6						X

Sum of Two Dice 1

- What is the most likely sum if you roll two dice?
- Pretend that one cube is yellow and the other is blue.

	Y-1	Y-2	Y-3	Y-4	Y-5	Y-6
B-1	2	3	4	5	6	7
B-2	3	4	5	6	7	8
B-3	4	5	6	7	8	9
B-4	5	6	7	8	9	10
B-5	6	7	8	9	10	11
B-6	7	8	9	10	11	12

- The probability of getting 7 is $6/36 = 1/6$.
- If the dice are identical, we have a non-uniform probability model. The probability of getting 7 is $3 \cdot 1/18 = 1/6$, while the probability of getting 6 is $2 \cdot 1/18 + 1 \cdot 1/36 = 5/36$.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2		4	5	6	7	8
3			6	7	8	9
4				8	9	10
5					10	11
6						12

Sum of Two Dice 2

- This is important in many board games, for example in Catan.

Four Types of Sampling

- Select k from n objects

	With Replacement	Without Replacement
Ordered	n^k	$\frac{n!}{(n-k)!} = n \cdots (n-k+1)$
Unordered	$\binom{k+n-1}{k}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

- Note that $\frac{n!}{k!(n-k)!} = \frac{n!/(n-k)!}{k!} = \frac{n \cdots (n-k+1)}{1 \cdots k}$.
- $\binom{k+n-1}{k}$ can be written as $\left(\!\binom{n}{k}\!\right)$ and is called multichoose.
- Which of these four numbers is the largest?
- Ordered with replacement largest, unordered without replacement smallest.

Stars and Bars 1

- Suppose you select $k = 5$ from $n = 3$ objects, unordered with replacement. If we denote the objects by the numbers 1, 2 and 3, then we can denote our samples by for example, $[1, 1, 2, 2, 3]$, $[1, 1, 1, 1, 1]$, $[1, 1, 3, 3, 3]$. (These are called multisets.)
- Imagine a pill box with 3 compartments. Let [be the end walls, | the partitions (bars) and $*$ the pills. Then the selections above can be described as $[* * | * * | *]$, $[* * * * * ||]$, $[* * || * **]$.
- Now imagine a board with $k + n - 1 = 7$ holes, where each hole should be filled with a star or a bar. Such a location is determined by where you place the k stars (or the $n - 1$ bars).
- $\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$. (Remember that $k + n - 1 - k = n - 1$.)

Stars and Bars 2

- Another way to prove this formula, is to assume that we are choosing among the numbers $\{1, \dots n\}$, and observe that an unordered sample with repetitions can be described as a list of numbers

$$1 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq n.$$

If we now set $y_i = x_i + i - 1$, we get a list of numbers

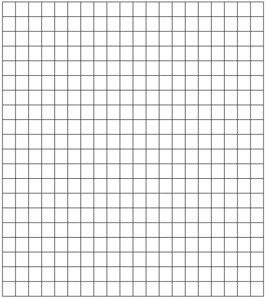
$$1 \leq y_1 < y_2 < \cdots < y_k \leq n + k - 1,$$

and the number of such lists is $\binom{n+k-1}{k}$.

Painting Dice

- You want to paint k dice, and you have n different colors. How many ways can you do this?
- Are the dice identical? Can two dice have the same color?
- Assume that the dice are identical and that two dice can have the same color. Then the answer is $\binom{n+k-1}{k}$.

Tables

- There are 20 teams in the English Premier League. How many games are there in a season? Use a table!
- 
- $20 \cdot 20 - 20 = 380 = 20 \cdot 19$. Ordered sampling without replacement.

Two Samples with Table 1

- There are three girls and two boys in a class. You choose two students. What is the probability that you choose two girls?
- Discuss: Describe four different ways of choosing two students, and determine the corresponding probabilities. You may want to use this table.

	G1	G2	G3	B1	B2
G1					
G2					
G3					
B1					
B2					

Two Samples with Table 2

	G1	G2	G3	B1	B2
G1					
G2					
G3					
B1					
B2					

- Prize for the best at mathematics and a different prize for the best at English. Ordered with replacement, $9/25$, $25 = 5^2$. The whole table.
- Prize for the best at mathematics and an identical prize for the best at English. Unordered with replacement, $6/15$, $15 = (6 \cdot 5)/(1 \cdot 2)$. On or above the diagonal.
- First and second prizes for the two who are best at mathematics. Ordered without replacement, $6/20$, $20 = 5 \cdot 4$. Above or below the diagonal.
- Two identical prizes for the two who are best at mathematics. Unordered without replacement, $3/10$, $10 = (5 \cdot 4)/(1 \cdot 2)$. Above the diagonal.
- Discuss: What is wrong with one of these arguments?

Two Samples with Table 3

- I explained it like once, but on my way home, I suddenly realized that it was wrong! I had made the same mistake that I had warned about in the beginning of my talk! (If you think you understand probability, you don't!)
- For unordered with replacement, we get a non-uniform distribution, just like in the case of the two dice. We therefore cannot use favorable/possible.

Two Samples with Table 4

- To count the *number* of ways we can select two girls, it matters whether the sampling is ordered or unordered, but if we want to find the *probability* that we select two girls, it does not matter role whether the sample is ordered or unordered!
- Unordered sampling reduces the number of possible outcomes, by merging the outcomes below the diagonal with the outcomes above the diagonal, and doubling the probability of these outcomes.
- So the probability of the simple event "choose girls 1 and 2" in the unordered model is equal to the probability of the compound event in the ordered model that consists of the union of the two simple events "choose girl 1 in round 1 and choose girl 2 in round 2" and "choose girl 2 in round 1 and choose girl 1 in round 2".

Two Samples with Table 5

- For unordered, the answer is therefore *not* $6/15$. The 5 outcomes along the diagonal have a probability of $1/25$ each, while the 10 over the diagonal have a probability of $2/25$ each. The probability for two girls is therefore $3 \cdot 1/25 + 3 \cdot 2/25 = 9/25$, which is the same as the probability for ordered with replacement.
- If there is no replacement, there is nothing on the diagonal, so we have uniform models, and the two answers, $6/20$ and $3/10$, are equal.
- This example shows that we must be humble in the face of probability. But it is all the more fun when we understand it!

Poker Hands 1

In poker, an ace can be high (14) or low (1), but not both at the same time. Both A-2-3-4-5 and 10-J-Q-K-A are straights, while Q-K-A-2-3 is not. However, the suits are not ranked, so 10-J-Q-K-A in Spades and 10-J-Q-K-A in Hears are tied.

Straight flush A straight flush is determined by its highest-ranking card. They go from 5 (A-2-3-4-5) to A (10-J-Q-K-A) in each of the four suits.

$$\binom{10}{1}\binom{4}{1} = 40.$$

Four of a kind Any one of the thirteen ranks can form the four of a kind by selecting all four of the suits in that rank. The final card can have any one of the twelve remaining ranks, and any suit.

$$\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1} = 624.$$

Poker Hands 2

Full house A full house comprises a triple (three of a kind) and a pair. The triple can be any one of the thirteen ranks, and consists of three of the four suits. The pair can be any one of the remaining twelve ranks, and consists of two of the four suits.

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3,744.$$

Flush The flush contains any five of the thirteen ranks, all of which belong to one of the four suits, minus the 40 straight flushes.

$$\binom{13}{5}\binom{4}{1} - 40 = 5,108.$$

Poker Hands 3

Straight The straight consists of any one of the ten possible sequences of five consecutive cards, from 5-4-3-2-A to A-K-Q-J-10. Each of these five cards can have any one of the four suits. Finally, as with the flush, the 40 straight flushes must be excluded.

$$\binom{10}{1}\binom{4}{1}^5 - 40 = 10,200.$$

Three of a kind Any of the thirteen ranks can form the three of a kind, which can contain any three of the four suits. The remaining two cards can have any two of the remaining twelve ranks, and each can have any of the four suits.

$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54,912.$$

Poker Hands 4

Two pair The pairs can have any two of the thirteen ranks, and each pair can have two of the four suits. The final card can have any one of the eleven remaining ranks, and any suit.

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123,552.$$

Pair The pair can have any one of the thirteen ranks, and any two of the four suits. The remaining three cards can have any three of the remaining twelve ranks, and each can have any of the four suits.

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1,098,240.$$

Poker Hands 5

High card High card is a hand that does not fall into one of the above categories; i.e., the complement of the union of all the above hands. The number of such hands can be found by subtracting all the numbers above from $\binom{52}{5}$.

However, we can find the number directly, since a high card hand is a hand without a pair that is not a straight or a flush. So it contains five of the thirteen ranks, discounting the ten possible straights, and each card can have any of the four suits, discounting the four possible flushes.

$$\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right] = 1{,}302{,}540.$$

High Card 7 Paradox

A pair of Queens beats a pair of sevens. This is an arbitrary, but logical convention, since both hands are equally hard to get.

In the same way high card queen beats high card seven.

This is however, not logical, since high card 7 is much harder to get.

You must have five different values between 2 and 7, skipping 3, 4, 5 or 6, and not all of the same color.

$$\binom{4}{3}\left[\binom{4}{1}^5 - 4\right] = 4,080.$$

This is the worst hand in poker, but “should” be between full house and flush.

Simpson's Paradox in University Admission 1

- In 1973, Univ. of California Berkeley admitted 44% of males and 35% of females who applied for graduate school. This looked like a clear case of gender discrimination.
- However, the data from the six largest departments gave a different picture.

Department	Male acceptance rate	Female acceptance rate
A	62%	82%
B	63%	68%
C	37%	34%
D	33%	35%
E	28%	24%
F	6%	7%

Simpson's Paradox in University Admission 2

- Here is a table with the number of applicants included.

	Male		Female	
	Applicants	%	Applicants	%
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	373	6%	341	7%

- Many women apply to departments with low acceptance rates, while many men apply to departments with high acceptance rates.
- A trend appears in different groups of data but disappears or reverses when these groups are combined.

Simpson's Paradox in Grades

- You are teaching a class with 20 regular students and 5 students with various types of learning disabilities. The regular students score 80 out of 100 on a test, and the other students score 50 out of 100. The class average is 74 out of 100.
- Your Principal is impressed, and next year she gives you 10 students with learning disabilities. This year the average score among the regular students increase to 82 and the score for the other students increase to 52. You are very happy, but your Principal is not. Why?
- The average score decreased to 70!

Simpson's Paradox in Salaries

- You have a sample of 20 male and 5 female engineers, and 15 male and 10 female teachers. The average salaries among the four groups are as follows.

	Men	Women
Engineers	$ 80,000	$86,000
Teacher	$40,000	$43,000
Average	$62,857	$57,333

Boys and Girls 1

- Imagine a society that has a cultural preference for boys, and suppose all families continue having babies until they get a boy, at which time they stop. Suppose for simplicity that you are not allowed to have more than 4 children. (This last condition is actually not significant. We ignore identical twins and other complications, and assume equal and independent birthrates.)
- Discuss: In this society, will there be
  1. More boys that girls?
  2. More girls than boys?
  3. Equally many boys and girls?

Boys and Girls 2

- The simple answer is that the numbers must be equal. Wanting boys does not make more boys. The only way you can skew the numbers are through selective abortions or killing baby girls.
- Expected number of boys:
  $1 \cdot 1/2 + 1 \cdot 1/4 + 1 \cdot 1/8 + 1 \cdot 1/16 = 15/16$,
  Expected number of girls:
  $1 \cdot 1/4 + 2 \cdot 1/8 + 3 \cdot 1/16 + 4 \cdot 1/16 = 15/16$.

children	boys	girls	probability
1	1	0	1/2
2	1	1	1/4
3	1	2	1/8
	1	3	1/16
4	0	4	1/16

The Birthday Problem

- Suppose you have n people. How likely is it that at least two people have the same birthday?
- Most probability questions involving an "at least" statement is computing by using the formula

$$P(\text{at least two occurrences}) = 1 - P(\text{no occurences}), \text{so}$$
$$P(\text{at least two common birthdays})$$
$$= 1 - P(\text{no common birthdays}).$$

- The probability of n different birthdays is:

$$\frac{365}{366}\frac{364}{366}\cdots\frac{366-(n-1)}{366}$$
$$= \left(1 - \frac{1}{366}\right)\left(1 - \frac{2}{366}\right)\cdots\left(1 - \frac{n-1}{366}\right).$$

When $n = 23$, this dips below $1/2$.

Benford's Law

- Benford's Law states that in "naturally occurring" sets of numbers, the probability that the first digit is d is $\log_{10}(1 + 1/d)$.

digit	1	2	3	4	5	6	7	8	9
%	30.1	17.6	12.5	9.7	7.9	6.7	5.8	5.1	4.6

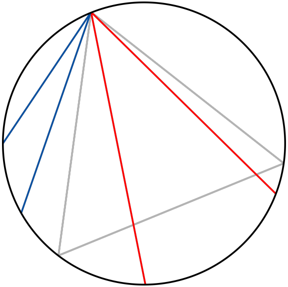
- Imagine a stock index that starts at 100. To get to a first digit of 2, the index must increase to 200, a 100% increase. Assume that the index goes up at a rate of about 10% a year. That means that it would take seven years to go from 1 to 2 as a first digit.
- But suppose we start at 900. It then takes only a little more than a year to reach 1,000 and a first digit a 1.

Bertrand’s Circle Paradox 1

- In 1899, Joseph Bertrand introduced the following paradox. Consider an equilateral triangle inscribed in a circle, and choose a chord at random. What is the probability that the chord is longer than a side of the triangle? Bertrand gave three arguments, all apparently valid, yet yielding different results.

Bertrand's Circle Paradox 2

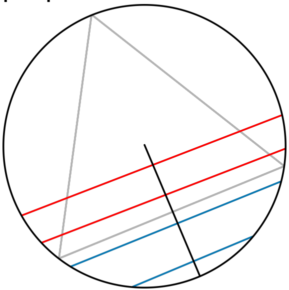
- Random endpoints: Choose two random points on the circle and draw the chord joining them. We can rotate the triangle so that a vertex coincides with one of the



endpoints of the chord.
- If the other endpoint lies on the arc between the other vertexes, then the chord is longer than the side.
- The length of the arc is one third of the circumference, so the probability that the chord is longer than the side is $1/3$.

Bertrand's Circle Paradox 3

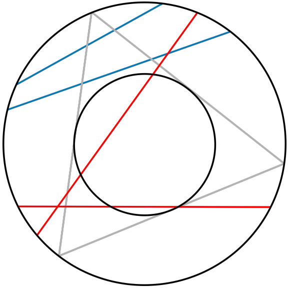
- Random radius: Choose a radius and a point on the radius and construct the chord through the point perpendicular to the radius. We can rotate the triangle so that a side is perpendicular to the radius.



- The chord is longer than the side of the triangle if the point is nearer the center of the circle than the point where the side intersects the radius.
- The side of the triangle bisects the radius, so the probability that the chord is longer than the side is $1/2$.

Bertrand's Circle Paradox 4

- Random midpoint: Choose a point anywhere within the circle and construct a chord with the point as its midpoint.



- The chord is longer than the side of the triangle if the point is within a the inscribed circle of triangle, which has radius $1/2$ the radius of the larger circle.
- The area of the inscribed circle is one fourth the area of the circumscribed circle, so the probability that the chord is longer than the side is $1/4$.

Bertrand's Circle Paradox 5

- What is the right answer? In 1973, Edwin Jaynes argued that random midpoint and $1/2$ was the correct answer, but this has since been disputed. For more details, please see `https://en.wikipedia.org/wiki/Bertrand_paradox_(probability)`.
- The key point is that continuous probabilities are hard, and depends on how you measure infinite sets.

A Family of Paradoxes

- There is a family of paradoxes that all have similar calculations.
  - The Monty Hall Problem (Car and Goat),
  - The Two-Child Paradox,
  - Bertrand's box paradox,
  - Three Prisoners,
  - Principle of Restricted Choice in Bridge.

The Monty Hall Problem (Car and Goat)

- You are on a game show and you are faced with three doors. Behind one of them is a car and you are asked to pick a door, and you choose door 1. Before the host opens it, she first opens another door, and you see that there is nothing behind it. She then asks you if you are sure about door 1, or if you want to switch your guess.
- Suppose you pick door 1 and that the host opens door 2. The probability that the car is behind 1 is $1/3$, so you are probably wrong, but after the host opened door 2, you know that if the car is not behind 1 (and it probably is not), then it must be behind 3, so you should switch!
- We must assume that if the host has a choice, they open a door at random. If the host usually opens door 2 when I open door 1, then seeing that the host opened door 3, would give me extra information.

The Two-Child Paradox 1

- A man has two children. (We ignore identical twins and other complications, and assume equal and independent birthrates.) You know, in various ways, that he has at least one boy. What is in each case the probability that both are boys?
  1. You know that the oldest child is a boy.
  2. You know that at least one of the children is a boy.
  3. You ask the man if at least one of the children is a boy and he says yes.
  4. You meet the man on the street with a boy. (I am assuming that there is no bias, i.e., it is equally likely that you see the boy as the girl if he has one of each.)
  5. You ask the man if at least one of the children is a boy born on a Tuesday and he says yes.
  6. You meet the man on the street on a Tuesday with a boy and are told he was born on a Tuesday.

The Two-Child Paradox 2

- Question 1 is simple. If the oldest child is a boy, then the probability that both are boys is $1/2$.
- In fact, this is the case whenever you know that a “specific” child is a boy. So the answer to Questions 4 and 6 is also $1/2$, since in both cases you have picked a specific child.
- Question 2 is problematic, since we do not know what “know” means.
- Questions 3 and 5 are more difficult, and we will look at them carefully. The answers turn out to be quite surprising!

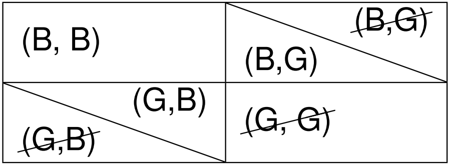
The Two-Child Paradox 3

- Look at the sample space of all families with two children, $\{(G,G),(G,B),(B,G),(B,B))\}$. (Note that I use ordered pairs, to get a uniform probability model.)
- Suppose you sample *families* and ask if the family has at least one boy. If so, it becomes part of the sample space. If not, delete it. The sample space becomes $\{(B,B),(B,G),(G,B)\}$ with a uniform distribution, and the probability of (B,B) is $1/3$, which answers Question 3.
- Suppose you instead sample *children* from families with two children and see if it is a boy. If so, it becomes part of the sample space. If not, delete it. The sample space becomes $\{(B,B),(B,G),(G,B)\}$ but now there is no uniform distribution, because (B,B) has twice the probability as (B,G) and (G,B) for half of these families have been deleted. So now the probability of (B,B) is actually $1/2$, which confirms our answer to Question 4.

The Two-Child Paradox 4

- However, this argument gives geometric insight, which will be useful later on.
-

(B, B)	(B,G)
(G,B)	~~(G, G)~~



The Two-Child Paradox 5

- We will now give a more general argument. Set $q = P(\text{You say he has at least one boy} \mid \text{He has } \{G, B\})$.
- If you sample the family, that is, you ask the father or otherwise have full information about the family, then $q = 1$.
- If you sample children, that is, you observe a boy, without knowing anything about the other child, then $q = 1/2$. You will miss half of the $\{G, B\}$ families, because half the time you see a girl.
- If there is a bias, like if the man spends more time with one child, then q may be different from $1/2$.

The Two-Child Paradox 6

- We use Bayes' formula.

$$P(\{G,B\}) \mid \text{At least one boy}) = \frac{P(\text{At least one boy} \mid \{G,B\})P(\{G,B\})}{P(\text{At least one boy})}.$$

-

$$\begin{aligned} &P(\text{At least one boy}) = \\ &P(\text{At least one boy} \mid \{G,B\})P(\{G,B\}) \\ &+P(\text{At least one boy} \mid \{G,G\})P(\{G,G\}) \\ &+P(\text{At least one boy} \mid \{B,B\})P(\{B,B\}). \end{aligned}$$

- $P(\{G,B\} \mid \text{At least one boy}) =$ $(q \cdot 1/2)/(q \cdot 1/2 + 1 \cdot 1/4) = 2q/(1+2q)$.
- It follows that $P(\{B,B\} \mid \text{At least one boy}) = 1-2q/(1+2q) = 1/(1+2q)$.
- $q = 1/2$ gives $P = 1/2$, and $q = 1$ gives $P = 1/3$.

The Two-Child Paradox 7

- We can illustrate this by the following variation. Suppose you know that I have either ten boys, or a boy and nine girls. You see me with a boy. Is it then most likely that there are nine boys or nine girls at home?
- If I have nine girls, you will probably see me with one of the girls, so if you see me with a boy, it is probably one of the ten boys.

The Two-Child Paradox 8

- We will now consider the case of the boy born on a Tuesday, or in general, a boy satisfying a condition with probability p. We will denote a boy that satisfies our condition by $B+$ or $B-$ otherwise.

(B+, B+) $p^2/4$	(B+, B-) $p/4 - p^2/4$	(B+,G) $p/4$
(B-, B+) $p/4 - p^2/4$	~~(B-, B-)~~ $1/4 - p/2 + p^2/4$	~~(B-,G)~~ $1/4 - p/4$
(G,B+) $p/4$	~~(G, B-)~~ $1/4 - p/4$	~~(G, G)~~ $1/4$

The Two-Child Paradox 9

- It follows from the table that the probability of two boys is

$$\frac{p/2 - p^2/4}{p/2 - p^2/4 + p/2} = \frac{p-2}{p-4}.$$

- Notice that if $p = 1$, i.e., we do not get any new information, then the probability is $1/3$, just as if we were just told that he had at least one boy. However, if p is close to 0, i.e., it is a rare condition that gives us a lot of information, then the probability is close to $1/2$.
- Geometrically, we observe that the sum of the first two cells in the first column equals the last cell, and same for the first row. However, the probability is less than $1/2$ because the top-left cell cannot be counted twice.
- If you have two boys, it is more likely that you will have a $B+$ boy, and conversely if you have a $B+$ boy, it is more likely that it is because you have two boys.

Bertrand's Box Paradox 1

- I have three cards, one that is black on both sides, one that is white on both sides, and one that is black on one side and white on the other. Discuss:
  1. I show you one card and you see a black side. What is the probability that the other side it black, too?
  2. I select a card, and tell you that it has at least one black side. What is the probability that it has two black sides?

Bertrand's Box Paradox 2

- There are two ways you can see a black side. Either because I picked the BB card, or because I picked the BW card, and then showed you the B side. However, if I picked the BW card, I would half the time end up showing you the W side, so if you see a black side, it will $2/3$ of the time be because it is the BB card.
- However, in the second card, I will tell you that the card has at least one black side both if it is the BB or the BW card, so the probability that it is the BB card is $1/2$.

Three Prisoners 1

- There are three prisoners on death row, A, B and C. The Governor has decided to pardon one of them. You are the Warden, and you know who will be pardoned. A asks you if you give the name of one of the two others who will be hanged.
- You think about it, and say that B will be hanged. A is now happy, believing that the probability that they will be pardoned has increased from $1/3$ to $1/2$. Discuss: Is this right?

Three Prisoners 2

- I C is pardoned, then the Warden does not have a choice, and must say that B will be hanged. However, if it is you who will be pardoned, then the Warden has a choice. Half the time they will say that B will be hanged and half the time that C will be hanged.
- So if you are told that B will be hanged, it will $2/3$ of the time be because C is pardoned, and only $1/3$ of the time because you are pardoned.
- Again we are assuming that if the Warden has a choice, then they will choose randomly. If they have a bias towards choosing B rather than C when they have a choice, i.e., when you are pardoned, then the computations are more complicated.

Principle of Restricted Choice in Bridge 1

- You play a notrump contract as South and you have ♠8 7 5 4 while your “dummy” partner, whose cards are face down on the table and visible to all, has ♠A J 10 9 6. So you know that West and East have ♠K Q 3 2. How are these cards split between E and W?

Principle of Restricted Choice in Bridge 2

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Split	Probability	Hands	Prob. of each hand
$2-2$	$\binom{4}{2}\binom{22}{11}/\binom{26}{13} = 40.7\%$	6	6.8%
$3-1$	$2\binom{4}{1}\binom{22}{12}/\binom{26}{13} = 49.7\%$	8	6.2%
$4-0$	$2\binom{4}{0}\binom{22}{13}/\binom{26}{13} = 9.6\%$	2	4.8%

Principle of Restricted Choice in Bridge 3

- You now play a small spade, and W plays ♠2. You hope that W has the K and the Q, so you play ♠6 from the dummy. However, E takes the trick with ♠K. You later on win another trick, and play another small spade, and W plays ♠3. Where is the queen?
- Either W had ♠Q 3 2 and E had ♠K or W had ♠ 3 2 and E had ♠KQ. In the first case, E did not have a choice, and had to play the king. In the second case, E had a choice between playing the King or the Queen, and chose to play the King, since the King and the Queen are equivalent.
- The previous table shows that the 2-2 split is marginally more likely than the 3-1 split, but the main point is that the Principle of Restricted Choice says that E probably did not have a choice.

Principle of Restricted Choice in Bridge 4

- Let $q = P(\text{E plays K} \mid \text{E has KQ})$. Then

$$P(\text{E has KQ} \mid \text{E plays K}) = \frac{P(\text{E plays K} \mid \text{E has KQ})P(\text{E has KQ})}{P(\text{E plays K})}.$$

- Now

$$\begin{aligned} P(\text{E plays K}) &= P(\text{E plays K} \mid \text{E has KQ})P(\text{E has KQ}) \\ &\quad + P(\text{E plays K} \mid \text{E has K})P(\text{E has K}) \\ &= q \cdot 0.068 + 0.062, \end{aligned}$$

so we get

$$P(\text{E has KQ} \mid \text{E plays K}) = \frac{q \cdot 0.068}{q \cdot 0.068 + 0.062} \approx \frac{q}{q+1}.$$

Principle of Restricted Choice in Bridge 5

- So if $q = 1/2$, then

$$P(\text{E has KQ} \mid \text{E plays K}) \approx \frac{1/2}{1/2 + 1} = \frac{1/2}{3/2} = \frac{1}{3}.$$

Two Aces Paradox 1

- I have two decks of cards. I give two cards to Alice from one, and two cards to Bob from the other. Alice says that she has at least one ace, and Bob says that he has the ace of spades.
- Discuss: Are they both equally likely to have two aces?

Two Aces Paradox 2

- The probability that Alice has two aces given that she has at least one ace is $P(2 \text{ aces} \mid \text{At least one ace}) = P(2 \text{ aces})/P(\text{At least one ace})$.
- There are $\binom{52}{2}$ possible hands consisting of two cards, $\binom{4}{2} = 6$ of them consists of two aces and $4 \cdot 48$ consists of exactly one ace. This gives

$$\frac{6/\binom{52}{2}}{(6+4\cdot 48)/\binom{52}{2}} = 6/(6+4\cdot 48) = 6/198 = 1/33 \approx 0.03.$$

(Note how the fraction of probabilities is the same as the fraction of frequencies, since I expand with the number of possible outcomes.)

Two Aces Paradox 3

- For Bob I get $P(\text{2 aces} \mid \text{Ace of spades}) = P(\text{Ace of spades and another ace})/P(\text{Ace of spades})$. Since 3 of them consist of ace of spades and one more ace, and 51 of them consist of ace of spades and one more card, this becomes $3/51 \approx 0.06$.
- The point here is that there are only about a quarter as many hands that contain the ace of spades as there are hands that contain at least one ace, while there are exactly half as many hands that contain the ace of spades and another ace as there are hands that contains two arbitrary aces. It is therefore about twice as likely that you have two aces if I know you have the ace of spades than if I only know that you have an unspecified ace.

Two Aces Paradox 4

- Another way to think of it is that if you have two aces, you have two chances of having the Ace of Spades. That means that most of the time you have the Ace of Spades, it will be because you have aces.
- Specifying the suit is a more restrictive condition on hands with one ace than on hands with two aces.

History of Probability

- Is history of mathematics useful for teachers and students?
- Yes, if you choose the right themes.
- The early history of probability is a very good theme.
- Both Luca Pacioli (ca. 1447–1517) (Summa de arithmetica, geometria, proportioni et proportionalita, Summary of arithmetic, geometry, proportions and proportionality, 1494) and Gerolamo Cardano (1501—1576) (Liber de ludo aleae, The Book on Games of Chance, ca. 1564) solved simple problems in probability, but when they tried to solve more difficult problems, it often went wrong.

History of Probability 2

- In 1654, Antoine Gombaud, Chevalier de Méré asked Blaise Pascal (1623–1662) two questions. The problems had been discussed by Pacioli and Cardano, but they were finally solved by Pascal in an exchange of letters with Pierre de Fermat (1601-1665), and this is considered the beginning of the study of probability.
- Is it true that the probability of getting at least one six in four rolls with one dice is over 50 percent, but that the probability of getting two sixes at least once in 24 rolls with two dice is less than 50 percent?

Chevalier de Méré's First Problem - Double Sixes

- Cardano thought he had already solved Chevalier de Méré's first problem. Since the probability of getting a six is 1/6, Cardano thought he would get a six every sixth time he rolled the dice. Therefore, there is a 50 % chance of getting a six on three throws according to Cardano.
- Since the probability of getting two sixes on a roll of two dice is 1/36, he thought that he would get two sixes every 36 times he rolled two dice. Therefore, there is a 50 % chance of getting two sixes on 18 throws according to Cardano.

Chevalier de Méré’s First Problem - Double Sixes 2

- Chevalier de Méré thought that Cardano was wrong, and that you had to roll four times with one dice to be able to bet on getting at least one six.
- This is correct, since the probability of not getting any sixes in four throws is $(5/6)^4 \approx 0,48$, while $(5/6)^3 \approx 0,58$.
- If you roll two dice, there are now 36 possible outcomes, and Chevalier de Méré thought he could use the same ratio $(4/6)$ as when rolling one dice, and that you could therefore expect to get at least a pair of sixes in $36 \cdot 4/6 = 24$ roll.
- This is wrong since $(35/36)^{24} \approx 0,51$, while $(35/36)^{25} \approx 0,49$.

Chevalier de Méré's Second Problem - The Problem of Points

- Two people play a game that consists of a series of rounds, and in each round they have an equal chance of winning. The winner is the one who has first won six rounds. How should the pot be divided if the game is interrupted when player A has won five rounds and player B has won two?
- Luca Pacioli thought we should look at how many games each had won. He thought that A should get 5/7 of the pot.
- This solution was criticized in 1556 by Niccolo Tartaglia (who solved the cubic equation). He looked at a game that was interrupted after only one round. Is it fair that A gets the whole pot because they lead 1-0?

Chevalier de Méré's Second Problem - The Problem of Points 2

- Tartaglia said that if A leads by three points, which is half the number needed to win, then A should get half of B's bet, so A gets $3/4$ and B $1/4$ of the pot.
- Is it reasonable for 5-2 and 3-0 to be treated equally?
- Cardano understood that you have to look at how many games the players need to win, not how many they have won. He tried to make an inductive argument, but did not succeed.

Chevalier de Méré's Second Problem - The Problem of Points 3

- Pascal assumes that the score is 5-4 and that the pot is 80. If A wins the next round, A has won, but if B wins the round, they are tied.
- It is therefore reasonable that the pot is divided in two, that A gets one half, and that they share the other half equally. So A gets 60 and B gets 20.
- Now assume that the score is 5-3. Again we divide the pot into two equal parts. If A wins the next round, A wins the game, and if B wins the next round, the score is 5-4. But that is the case we looked at above! We therefore divide the pot into two equal parts, A gets one part, and the other part is divided 3 to 1.
- Pascal then provides a complete solution involving Pascal's triangle and induction proof.

Chevalier de Méré's Second Problem - The Problem of Points 4 (Optional)

- Assume that A needs to win r rounds and that B needs to win s rounds. Set $n = r + s - 1$, and assume we are playing n rounds. Then one, and only one player will have won. A should then have $\sum_{k=0}^{s-1} \binom{n}{k}/2^n$ of the pot, and B should have $\sum_{k=s}^{n} \binom{n}{k}/2^n$.
- Assume that $r = 1$ and $s = 2$. Then the outcomes will be (A, A), (A, B), (B, A) and (B, B). A should have $(1 + 2)/4 = 3/4$ and B should have $1/4$.
- I could have said that the outcomes were (A), (B, A) and (B, B), but then the sample space would not have been uniform.

Chevalier de Méré's Second Problem - The Problem of Points 5 (Optional)

- If $r = 2$ and $s = 5$, A must have

$$\left(\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4}\right) \Big/ 2^6$$

and B shall have

$$\left(\binom{6}{5} + \binom{6}{6}\right) \Big/ 2^6.$$