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Laws of Trigonometry from an Advanced Viewpoint

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School Mathematics from an Advanced Standpoint

- At the Univ. of Oslo, I teach a course called MAT4010 School Mathematics from an Advanced Standpoint. I also give talks based on material from this course, and today's lecture is part of that series.
- It is a mixture of mathematics and mathematics education, that I call educational mathematics.
- I will talk about mathematical topics that I believe are relevant for teachers.
- Some topics you can take straight to the classroom.
- Some topics may help you answer questions that you may be asked occasionally by strong pupils.
- Some topics you will never discuss with any pupils, but knowing it will help your own understanding of the topics.
- There should be a line of sight back towards school mathematics.

What Will I Cover in This Talk? 1

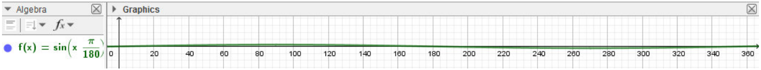
- I will start with a simple, but fundamental question: When and why do we use radians instead of degrees?
- Some students view the laws of trigonometry as a random list of examples. Could some other case come up on the exam, or have they seen all the possible cases?
- They have seen an ambiguous case. Are there other ambiguous cases?
- They have seen that the sine law sometimes does not work. When can you use the sine law safely?
- If the sine law gives you two solutions, can you then use the cosine law?
- Do you ever have a choice between the sine law and the cosine law? If you do, which one should you use?

What Will I Cover in This Talk? 2

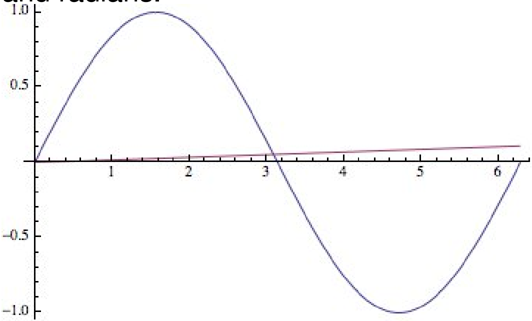
- I will present the laws of trigonometry in a systematic way based on congruence theorems.
- I will discuss the case of two sides and a non-included angle where we may get two solutions, and when we can safely use the law of sines.
- This last point often confuses people. For example, the Wikipedia page on Solution of triangles contains some unclear statements.
- I hope that this talk will give insights that will give you a better understanding of this important and interesting topic.

When and Why Do We Use Radians? 1

- This is the graph of $\sin(x)$, with x measured in degrees, and with the same scale along both axes.



- This is the graph of $\sin(x)$ with x measured in both degrees and radians.



When and Why Do We Use Radians? 2

- What is the derivative of $\sin(x)$ at the origin? You need to compute $\lim_{x \to 0} \sin(x)/x$ and if this limit is 1, then the derivative of $\sin(x)$ equals $\cos(x)$. This is true precisely if you measure x in radians, since then $\sin(x) \approx x$ if $x \approx 0$.
- However, if you use degrees, then the derivative equals $\pi/180 \approx 0.0175$.
- We use radians if and only if we are considering trigonometric functions.
- It is unnecceary snobbery to use radians when solving triangles.

Laws of Trigonometry from an Advanced Viewpoint 1

- My PhD thesis was called “Laws of Trigonometry on SU(3)”. SU(3) is a Lie group of dimension 8, but in this talk, we will just consider the Euclidean plane.
- However, at the end I will briefly describe how we can view the laws of trigonometry from an advanced viewpoint inspired by the work of the Norwegian mathematician Sophus Lie (1842–1899), and the Erlangen program of Felix Klein (1872).

Laws of Trigonometry from an Advanced Viewpoint 2

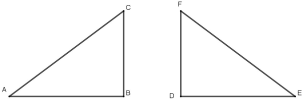
- Following Felix Klein, we will start by considering the group of isometries of the plane. I will discuss this in an informal way that does not require knowing any group theory.
- An isometry, also called a congruence transformation, is a transformation that preserves distance.
- Why are we interested in this?
- Because when solving triangles we are identifying triangles that are congruent. In mathematical terms, we are studying congruence classes of triangles under the group of congruence transformations.

Laws of Trigonometry from an Advanced Viewpoint 3

- There are two types of congruence transformations, direct and indirect (or opposite). Direct transformations preserve orientation, and indirect transformations reverse orientation.
- What are the direct congruence transformations of the plane?
- We can show that the group of direct congruence transformations are generated by translations and rotations.
- What are the indirect congruence transformations of the plane?
- They consist of reflections and glide reflections, i.e., compositions of reflections and translations.

Laws of Trigonometry from an Advanced Viewpoint 4

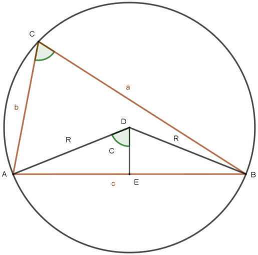
- Is this relevant for school mathematics?
- Are these triangles congruent?



- You have to make a choice. In this talk, I will consider them to be congruent, i.e., I use the full group of isometries.

The Extended Law of Sines

- It is easy to prove the following extension of the law of



  sines.
- Let R be the radius of the circumscribed circle. Since $\angle EDA = C$, we get

$$\sin C = \frac{c/2}{R}, \quad \text{so} \quad \frac{c}{\sin C} = 2R.$$

- Applying this to all the angles, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

The Importance of Notation

- I once saw a Norwegian secondary school teacher (grade 10) who did not know the convention of using the same letter for opposite sides and angles. That is no problem in grade 10, but many of his students will the next year study trigonometry, and if they are not familiar with this convention, the laws of trigonometry will be much harder to remember.

The Law of Cosines

- Notice that there are only two independent laws of sine. Why?
- If $\sin A/a = \sin B/b$ and $\sin B/b = \sin C/c$, then $\sin A/a = \sin C/c$, so the third law of sines is just a trivial combination of the two other.
- There are however three independent laws of cosines of the form

$$
\begin{aligned}
a^2 + b^2 - 2ab\cos C &= c^2 \\
a^2 + c^2 - 2ac\cos B &= b^2 \\
b^2 + c^2 - 2bc\cos A &= a^2.
\end{aligned}
$$

Laws of Trigonometry 1

- The law of sines requires that you know one “pair” of an angle and its opposite side, and one single side or angle. Think of the law of sines as a single friend going out with a couple.
- The law of cosines requires that you either know two sides and the included angle, and then find the last side, or that you know three sides and then find an angle, or that you know two sides and the angle opposite one of them. In the first two cases, you can think of the law of cosines as three single friends (either three sides, or two sides and a “single” angle) going out, while the last case is similar the law of sines.
- If you have a choice between using the law of sines or the law of cosines, the law of sines is easier to use computationally.

Solving a Triangle 1

- This raises several questions.
  1. When do you have a choice between using the law of sines or the law of cosines?
  2. When you do have a choice, which one should you use?
  3. If you know three sides, does it matter which angle you apply the law of cosines to?
  4. If you know three sides and one angle, does it matter which of the two other angles you apply the law of sines to?

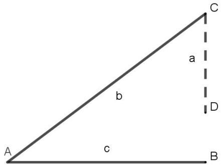
Solving a Triangle 2

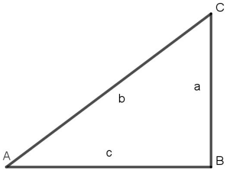
- We will now discuss the classical congruence theorems for triangles. We will not prove them, but give a complete list of such theorems.
- There are six quantities associated to a triangle: The three sides and the three angles.
- We will focus on the number of known sides.
- SSS: If we know three sides, we have the SSS (side-side-side) Theorem. We can only use the law of cosines.
- SAS: If we know two side and the angle between them, we have the SAS Theorem. We can only use the law of cosines.
- SSA: But what if we know two sides and the angle opposite one of them? In that case, we must distinguish between SsA, where the angle is opposite the longest of the two sides, and sSA where the angle is opposite the shorter of the two sides. We will discuss this later.

Solving a Triangle 3

- ASA: If we know only one side, we have the ASA Theorem. Remember that ASA really is AAAS, so it does not matter how the side is placed with respect to the two given angles. We can only use the law of sines
- AAA: If we do not know any sides, we have the AAA Theorem, but this is a similarity theorem, not a congruence theorem!
- Notice that two sides and the angle opposite one of them is the only case where we could use either the law of sines or the law of cosines. We will now look at that case in detail.

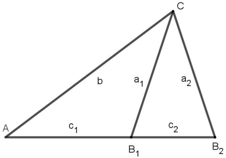
sSA and SsA 1

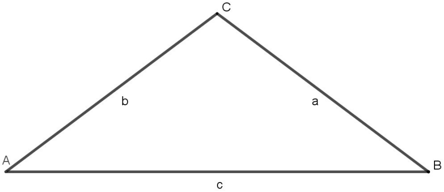
- Assume that we know the angle A and the sides a and b, and let us first consider the case when A is acute.
- 

- Since A is acute, the distance from C to the opposite side is $b \sin A$. If $a < b \sin A$, there will be no solutions.
- 

- If $a = b \sin A$, there will be one solution.

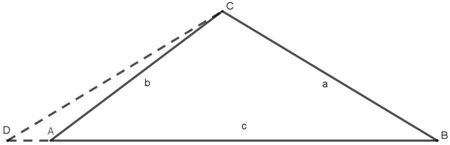
sSA and SsA 2

- 

- If $b \sin A < a < b$, there will be two solutions, since the circle centered at C with radius a will intersect the base line in two points.
- 

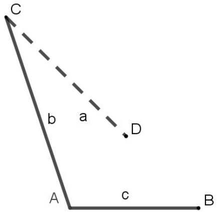
- If $a = b$, there will only be one solution, since the circle centered at C with radius a will intersect the base line in A and one point to the right, but intersection in A will not give us a triangle.

sSA and SsA 3

- 

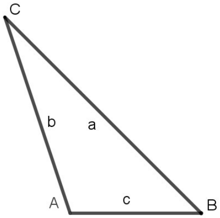
- If $a > b$, there will only be one solution, since the circle centered at C with radius a will intersect the base line in one point to the left of A and one point to the right, but the intersection to the left will give a triangle with different angles.

sSA and SsA 4

- Let us now consider the case when A is obtuse.
- 

- If $a < b$, there will be no solution.

sSA and SsA 5

- 

- If $a > b$, there will be one solution.
- We can conclude that SsA is a valid congruence theorem, while sSA will give us two triangles, except in the right angle case.

sSA and SsA 6

- In school, the ambiguous sSA case is usually discussed in conjunction with the law of sines.
- Sine is not injective on $[0^\circ, 180^\circ]$, and does not distinguish between acute and obtuse angles, so the law of sines may give two possible values.
- Some people get confused by this, and fail to understand that ambiguity means that there are two possible triangles, but instead seem to believe that ambiguity is simply some defect with the law of sines, which can be remedied by using the law of cosines.
- I hope that none of you suffer from this misconception, but for the sake of completeness, let us see explicitly why the law of cosines also suffers from the sSA problem.

sSA and SsA 7

- If you know three sides and use the law of cosines to determine an angle, you will get a unique value, since cosine is injective on $[0^\circ, 180^\circ]$.
- If you know two sides and the included angle, and use the law of cosines to determine the third side, you get a quadratic equation without a first order term, which therefore only has one positive solution.
- However, if you know two sides and the angle opposite one of them, and use the law of cosines to determine the third side, you get a quadratic equation with first order term, which may have two positive solutions.

sSA and SsA 8

- Let us now do this algebraically.

$$a^2 = x^2 + b^2 - 2xb\cos A$$
$$x^2 - (2b\cos A)x + b^2 - a^2 = 0$$
$$x = \frac{2b\cos A \pm \sqrt{4b^2\cos^2 A - 4(b^2 - a^2)}}{2}$$
$$x = b\cos A \pm \sqrt{b^2(\cos^2 A - 1) + a^2}$$
$$x = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$$

sSA and SsA 9

- A is acute, we will get two positive roots if

$$b\cos A - \sqrt{a^2 - b^2\sin^2 A} > 0$$
$$b^2\cos^2 A > a^2 - b^2\sin^2 A$$
$$b^2\cos^2 A + b^2\sin^2 A > a^2$$
$$b^2 > a^2$$
$$b > a.$$

sSA and SsA 10

- A is obtuse, then $b\cos A < 0$, so we will get no positive roots if

$$b\cos A + \sqrt{a^2 - b^2\sin^2 A} < 0$$
$$a^2 - b^2\sin^2 A < b^2\cos^2 A$$
$$a^2 < b^2\cos^2 A + b^2\sin^2 A$$
$$a^2 < b^2$$
$$b > a.$$

- We see that the law of cosines also breaks down in the sSA case.

Solving a Triangle 1

- However, there is no way around the fact that sine does not distinguish between acute and obtuse angles, so how can we use the law of sines safely?
- The secret to using the law of sines safely, is to never use it to try to find an angle that could be obtuse.
- Since there can be at most one obtuse angle in a triangle, we can instead say that you should not use the law of sines to try to find the largest angle in a triangle.
- If you know that the angle is not the largest, you are OK, since there can be at most one obtuse angle in a triangle.
- If the angle you are trying to find is opposite the shorter side, sine will not give you any problems, so SsA is a valid theorem, while sSA will not work.

Solving a Triangle 2

SSS Use the law of cosines for the *largest* angle (the angle opposite the *longest* side), use the law of sines for another angle, and then use the sum of the angles theorem. (Remember that cosine is a *long* word.)

SAS Use the law of cosines for the opposite side, the law of sines for the *smaller* of the two remaining angles, and the sum of the angles theorem. (Remember that sine is a *short* word.)

SsA Use the law of sines for the angle opposite the *short* side, use the sum of the angles theorem, and then the law of sines for the last side.

ASA Use the sum of the angles theorem, and then use the law of sines twice. Remember that [ASA] really is [AAAS], so it does not matter how the side is placed with respect to the two given angles.

Solving a Triangle 3

- This sometimes leads to statements like the following, taken from the Wikipedia page on Solution of triangles.

1. To find an unknown angle, the law of cosines is safer than the law of sines. The reason is that the value of sine for the angle of the triangle does not uniquely determine this angle. For example, if $\sin\beta = 0.5$, the angle β can equal either 30° or 150°. Using the law of cosines avoids this problem: within the interval from 0° to 180° the cosine value unambiguously determines its angle. On the

Solving a Triangle 4

- They continue with the following statement.

  Three sides given (SSS) [edit]

  Let three side lengths a, b, c be specified. To find the angles α, β, the law of cosines can be used:[3]

  $$\alpha = \arccos \frac{b^2 + c^2 - a^2}{2bc}$$

  $$\beta = \arccos \frac{a^2 + c^2 - b^2}{2ac}.$$

  Then angle $\gamma = 180^\circ - \alpha - \beta$.

  Some sources recommend to find angle β from the law of sines but (as Note 1 above states) there is a risk of confusing an acute angle value with an obtuse one.

- My point is that by paying attention to the size of the angles, you can use the law of sines safely.

Solving a Triangle 5

- There is one subtle difference between the use of the law of sines in SSS and sSA. If we use the law of sines to find the second angle in the SSS case, we have two possible candidates. However, since we know the ordering of the lengths of the sides, we also know the ordering of the size of the angles, so we can easily tell if we should pick the acute or the obtuse angle. So using the law of sines is possible, provided we are careful. Using the law of cosines is more complicated computationally, but will only give us one candidate.
- However, in the sSA case, there are in fact two triangles, so neither the law of sines nor the law of cosines could help us. As we have seen, they both fail, for different reasons.

Laws of Trigonometry from an advanced viewpoint 1

- You can associate six variables to a triangle, namely the three sides and the three angles. However, the congruence class of a triangle is determined by only three variables, namely SSS, SsA, SAS, or ASA. That means that there should be at least three equations relating the six variables.
- We have six such equations, namely the three laws of cosines, the two laws of sines and the sum of the angles theorem. Are there any relationships between these laws?
- It can be shown that you can deduce the two laws of sine from the three laws of cosines, but what about the other way?
- It can be shown that you can deduce the three laws of cosine from the two laws of sines and the sum of the angles theorem.
- So there are only three independent laws of trigonometry, and they reduce the six dimensional space to a three dimensional space of congruence classes of triangles.

Laws of Trigonometry from an advanced viewpoint 2

- We will deduce the three laws of cosine from the two laws of sines and the sum of the angles theorem. Start with

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$A + B + C = 180,$$

and get

$$\frac{c}{\sin C} = \frac{b}{\sin(A + C)} \quad \text{and} \quad \frac{c}{\sin C} = \frac{a}{\sin A},$$

and finally

$$c(\sin A \cos C + \cos A \sin C) = b \sin C$$
$$c \sin A = a \sin C.$$

Laws of Trigonometry from an advanced viewpoint 3

- We now divide by $\cos C$ and get

$$c(\sin A + \cos A \tan C) = b \tan C$$
$$\frac{c \sin A}{\cos C} = a \tan C,$$

which leads to

$$\frac{c \sin A}{b - c \cos A} = \tan C$$
$$\frac{c^2 \sin^2 A}{\cos^2 C} = a^2 \tan^2 C,$$

Laws of Trigonometry from an advanced viewpoint 4

- We now use the identity

$$1 + \tan^2 C = \frac{1}{\cos^2 C}$$

and combine the equations above to get

$$c^2 \sin^2 A \left(1 + \frac{c^2 \sin^2 A}{(b - c\cos A)^2}\right) = a^2 \frac{c^2 \sin^2 A}{(b - c\cos A)^2}.$$

This leads to

$$(b - c\cos A)^2 + c^2 \sin^2 A = a^2,$$

which implies

$$b^2 - 2bc\cos A + c^2\cos^2 A + c^2 \sin^2 A = a^2$$
$$a^2 = b^2 + c^2 - 2bc\cos A.$$

What is the Dimension of the Set of Congruence Classes of Triangles? 1

- I will skip the mathematical details here, and just try to give you an intuitive idea.
- Let us first consider a simple example. What is the dimension of the set of spheres around the origin in $\mathbb{R}^3$?
- The dimension of $\mathbb{R}^3$ is 3, and the dimension of a sphere is 2. I claim that the dimension of the set of spheres around the origin in $\mathbb{R}^3$ is therefore $3 - 2 = 1$.
- One way to see this geometrically is to say that each such sphere will intersect the positive x-axis in a point, so the positive x-axis parameterize the set of spheres around the origin in $\mathbb{R}^3$. It serves as a cross-section or moduli space for the set of spheres.

What is the Dimension of the Set of Congruence Classes of Triangles? 2

- This is essentially a complicated way of saying that a sphere around the origin is determined by its radius, but the point here is to make you believe the formula

$$\text{dimension of set of classes} \\ = \text{dimension of ambient space} - \\ \text{dimension of each of the classes.}$$

What is the Dimension of the Set of Congruence Classes of Triangles? 3

- I claim that the set of triangles is of dimension 6. Why?
- A triangle is simply a triple of points, and each point is a pair of coordinates, so we get six coordinates in total.
- But we are not interested in the set of triangles, we are interested in the set of congruence classes of triangles.
- How big is the group of congruence transformations? It is generated by translations and rotations. Translations are specified by a vector with two coordinates, and rotations are specified by the size of the angle, so we get three coordinates in total.
- If you pick a triangle, and consider the class of triangles that are congruent to it, the dimension of this class should therefore be three.
- The set of classes should therefore have dimension $6 - 3 = 3$.

What is the Dimension of the Set of Congruence Classes of Triangles? 4

- Thanks for coming! I hope you learned something useful!