Numbers

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1 Natural numbers

These notes are based on a project written by Astri Strand Lindbæck, Camilla Helvig og Pia Lindstrøm wrote for MAT4010 in 20014.

Let $\mathbb{N}$ denote the positive, natural numbers $\{1,2,3,\ldots\}$. Remember that positive means >0 and negative means <0, so nonnegative is not the same as positive, but means positive or zero.

2 Why is $(-1)(-1)=1$?

One way to understand this, is to say that multiplying by -1 is the same as "flipping" across zero on the number line, in which case, flipping twice does nothing. However, it can be instructive to also see an algebraic proof. Assume that we know how to multiply natural numbers, and that we want to extend this to integers. We want to do this in such a way that the following three properties are preserved.

1. Commutative $ab=ba$
2. Associative $(ab)c=a(bc)$
3. Distributive $a(b+c)=ab+ac$

Assume that $a,b\in\mathbb{N}$. We know that

$$a(-b)=(-b)+(-b)+\cdots+(-b)=-ab \tag{1}$$

by repeated addition, and to compute $(-a)b$, we use Equation (1) and commutativity to get

$$(-a)b=b(-a)=-ba=-ab. \tag{2}$$

We want to show that $(-1)(-1) = 1$, and to do that, we consider $(-1)(-1) - 1$ and use distributivity

$$\begin{aligned}
(-1)(-1) - 1 &= \\
(-1)(-1) + (-1) &= \\
(-1)(-1) + (-1) \cdot 1 &= \\
(-1)(-1+1) &= \\
(-1) \cdot 0 &= 0.
\end{aligned}$$

Hence $(-1)(-1) = 1$.

3 Why is $0.\bar{9} = 1$?

There are many ways to see that $0.\bar{9} = 1$, where $\bar{9}$ denotes an infinite string of 9's. Since $1/3 = 0.\bar{3}$,we can multiply by 3 and get $0.\bar{9} = 1$. We can also write

$$\begin{aligned}
x &= 0.\bar{9} \\
10x &= 9.\bar{9} \\
10x &= 9 + 0.\bar{9} \\
10x &= 9 + x \\
9x &= 9 \\
x &= 1
\end{aligned}$$

We can also use that $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ for $|x| < 1$ to show that

$$\begin{aligned}
0.\bar{9} &= 9(0.1 + 0.01 + 0.001 + \cdots) = 9(0.1 + 0.1^2 + 0.1^3 + \cdots) \\
&= 9 \sum_{k=1}^{\infty} 0.1^k \; = 9 \cdot 0.1 \sum_{k=0}^{\infty} 0.1^k = 0.9 \frac{1}{1-0.1} = \frac{0.9}{0.9} = 1.
\end{aligned}$$

Finally, we can argue that they have to be equal, since if they were not equal, there would have to be some number between them. However, there is no way to put any number between them.

In general, we claim that

$$a.a_1a_2 \ldots a_n = a.a_1a_2 \ldots (a_n - 1)\bar{9}, \tag{3}$$

where $a_n \neq 0$ and a can be positive, negative or zero. For instance

$$3.14 = 3.1(4-1)\bar{9} = 3.13\bar{9} \quad \text{and} \quad 0.11 = 0.1(1-1)\bar{9} = 0.10\bar{9}.$$

We can prove (3) as follows

$$
\begin{aligned}
a.a_1a_2\dots(a_n-1)\bar{9} &= a.a_1a_2\dots(a_n-1)+9(10^{-(n+1)}+10^{-(n+2)}+\cdots) \\
&= a.a_1a_2\dots(a_n-1)+9\sum_{k=n+1}^{\infty}0.1^k \\
&= a.a_1a_2\dots(a_n-1)+9\cdot 0.1^{n+1}\sum_{k=0}^{\infty}0.1^k \\
&= a.a_1a_2\dots(a_n-1)+9\cdot 0.1^{n+1}\frac{1}{1-0.1} \\
&= a.a_1a_2\dots(a_n-1)+0.1^{n+1}\frac{9}{0.9} \\
&= a.a_1a_2\dots(a_n-1)+0.1^{n+1}\cdot 10 \\
&= a.a_1a_2\dots(a_n-1)+0.1^{n} \\
&= a.a_1a_2\dots(a_n-1)+0.\overbrace{0\dots0}^{n-1}1 \\
&= a.a_1a_2\dots(a_n-1+1) = a.a_1a_2\dots a_n.
\end{aligned}
$$

So we see that every finite decimal expansion can also be written as an infinite decimal expansion. There is only exception, namely 0. One way to understand why 0 is exceptional is because for positive numbers, the infinite expansion “looks” smaller, wile for negative numbers, the infinite expansion “looks” bigger. So it is not surprising that 0 is a singular case.

4 Divison by zero

The key to understanding division and fractions is that

$$\frac{a}{b} = c \iff a = bc. \tag{4}$$

This shows why we cannot divide by 0. If $b=0$, we get

$$\frac{a}{0} = c \iff a = 0\cdot c = 0,$$

which shows that we get a contradiction if we try to assign a value to $a/0$ when $a \neq 0$. But what if $a=0$? In that case, the above equation just says that $0 = 0\cdot c = 0$, which is true for any c. But that is precisely the problem. We could theoretically define $0/0$ to be anything, without violating (4), but which value should we choose? Since we theoretically could pick any value, we say that $0/0$

is an indeterminate form. And even if we do not violate (4), could we get into other problems if we tried to pick a value for $0/0$? Consider the following equations

$$\begin{aligned} 0+0 &= 0 \\ 2\cdot 0 &= 1\cdot 0 \\ 2 &= 1\cdot\frac{0}{0}. \end{aligned}$$

This would give a contradiction if we tried to define $0/0$ to be anything other than 2, and if we tried to define it to be 2, we could just consider three zeros above, and get another contradiction. So we will simply say that $0/0$ is also undefined.

5 Divison by fractions

Many students do not understand why dividing by a fraction is the same as inverting the second fraction and multiplying as follows

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b}\frac{d}{c}. \tag{5}$$

To see this, we must show that if multiply the number on the right by the divisor, we get the divident, i.e.,

$$\left(\frac{a}{b}\frac{d}{c}\right)\frac{c}{d} = \frac{a}{b}\left(\frac{d}{c}\frac{c}{d}\right) = \frac{a}{b}.$$

Another way to see this, is to expand the fraction by the divisor as follows

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\frac{d}{c}}{\frac{c}{d}\frac{d}{c}} = \frac{\frac{a}{b}\frac{d}{c}}{1} = \frac{a}{b}\frac{d}{c}$$

or alternatively, to expand by the products of the denominators in the parts of the complex fraction as follows

$$\frac{\frac{a}{b}bd}{\frac{c}{d}bd} = \frac{ad}{bc}.$$

6 Powers

Assume that we have defined a^n with $n \in \mathbb{N}$ to be

$$a^n = \overbrace{a\cdot\ldots\cdot a}^{n} \tag{6}$$

For $n, m \in \mathbb{N}$ it is easy to see that we have the following property

$$a^n a^m = \overbrace{a \cdots a}^{n}\overbrace{a \cdots a}^{m} = \overbrace{a \cdots a}^{n+m} = a^{n+m}. \tag{7}$$

We now want to extend Definition (6) to $n \in \mathbb{Z}$ in way that preserves property (7). In other words, we will assume that (7) holds, and see what that implies about a^0 and a^{-n} for $n \in \mathbb{N}$. Setting $m = 0$ in (7), we get

$$a^n = a^{n+0} = a^n \cdot a^0,$$

so if $a \neq 0$, we can divide by a^n and conclude that $a^0 = 1$. (We will discuss 0^0 later.) We now set $m = -n$ in (7) and get

$$1 = a^0 = a^{n-n} = a^n a^{-n},$$

so it follows that

$$a^{-n} = \frac{1}{a^n}.$$

We can now check that

$$\frac{a^n}{a^m} = \frac{a^{n-m}a^m}{a^m} = a^{n-m}$$

holds for $n, m \in \mathbb{Z}$.

7 Fractional Exponents

Again we want to extend a definition to a larger set of numbers by preserving a property. We know that for $m, n \in \mathbb{N}$ we have

$$(a^n)^m = \underbrace{a^n \cdots a^n}_{m} = \underbrace{(\overbrace{a \cdots a}^{n}) \cdots (\overbrace{a \cdots a}^{n})}_{m} = \overbrace{a \cdots a}^{n \cdot m} = a^{n \cdot m} \tag{8}$$

We want to extend the definition of a^n to $n \in \mathbb{Q}$, while maintaining property (8). To see how to do this, we simply write $x = a^{1/n}$. Then

$$x^n = (a^{1/n})^n = a^{\left(\frac{1}{n}n\right)} = a^1 = a$$

so we see that

$$a^{1/n} = \sqrt[n]{a}.$$

Using property (8) again, we get that

$$a^{m/n} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}.$$

8 Is $0^0 = 1$?

We have seen that for $a \neq 0$ we have $a^0 = 1$, so $\lim_{a\to 0} a^0 = 1$. It therefore seems natural to define $a^0 = 1$. However, for $x > 0$, we have $0^x = 0$ and it follows that $\lim_{x\to 0^+} 0^x = 0$. This shows that the function $f(x,y) = x^y$ does not have a limit at $(0,0)$ since we get different values depending on how we approach $(0,0)$. It follows that f is not continuous at $(0,0)$. That makes it harder to find a good value for 0^0, but not impossible. The floor function $f(x) = \lfloor x \rfloor$, which denotes the greatest integer less than or equal to x, is not continuous at $x = 0$, but that does not prevent us from saying that $\lfloor 0 \rfloor = 0$.

It is safe to say that x^0 comes up more often that 0^x. If we write a polynomial like

$$p(x) = \sum_{k=0}^{n} a_k x^n, \quad \text{then} \quad p(0) = a_0 0^0 + \ cdots + a_n 0^n,$$

and we are implicitly assuming that $0^0 = 1$. We can also consider a power series like

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Then

$$f(0) = 1 = \sum_{n=0}^{\infty} 0^n = 0^0.$$

If we do not define 0^0 to be 1, we will have trouble with even simple expressions like this.

Another example is the Binominal Theorem

$$(a+b)^x = \sum_{k=0}^{\infty} \binom{x}{k} a^k b^{x-k}.$$

Setting $a = 0$ on both sides and assuming $b \neq 0$ we get

$$b^x = (0+b)^x = \sum_{k=0}^{\infty} \binom{x}{k} 0^k b^{x-k} = \binom{x}{0} 0^0 b^x = 0^0 b^x,$$

where, we have used that $0^k = 0$ for $k > 0$, and that $\binom{x}{0} = 1$. On the right hand side we have 0^0 and if we do not use $0^0 = 1$ then the binomial theorem (as written) does not hold when $a = 0$ because then b^x does not equal $0^0 b^x$.

Another reason why $0^0 = 1$ is because x^0 is the "empty product", which should 1. For the same reason $0! = 1$.

But we have to be careful with 0^0. We cannot compute it with like other numbers. We cannot say that

$$1 = 0^0 = 0^{n-n} = \frac{0^n}{0^n} = \frac{0}{0}.$$

So to sum up, we can write $0^0 = 1$ in sums, but we cannot apply the power rules to it.