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Calculus and Counterexamples

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Source of counterexamples

- $$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
- $$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Monotonicity

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$
- $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b).
- $f' \geq 0$ on $(a, b) \implies f$ is increasing on (a, b).
- $f' \geq 0$ on $(a, b) \impliedby f$ is increasing on (a, b).
- $f(x) = x^3$ shows that $f' \geq 0$ on $(a, b) \not\Longleftarrow f$ is strictly increasing on (a, b).

Extreme point 1

- If c is an extreme point and $f'(c)$ exists, then $f'(c) = 0$.
- First Derivative Test: If f' exists around c, and f' changes sign at c, then c is an extreme point.
- Second Derivative Test: If $f'(c) = 0$ and $f''(c)$ is positive (negative), then c is a minimum (maximum).

Extreme point 2

- If f' changes sign at c, then c is an extreme point. The converse is not always true.
- $f(x) = x^2(2 + \sin(1/x))$,
  $f'(x) = 4x + 2x\sin(1/x) - \cos(1/x)$.
- $x^2 + x^2\sin(1/x))$ has infinitely many zeros.
- If f' is positive on (a, b), then f is increasing on (a, b). But what if we only know that $f'(c) > 0$? Can we say that f is increasing on an interval around c?
- $f(x) = x + 2x^2\sin(1/x)$,
  $f'(x) = 1 + 4x\sin(1/x) - 2\cos(1/x)$ is both positive and negative in every neighborhood of 0.

Point of inflection

- We say that c is a point of inflection if f'' changes sign at c and f has a tangent line at c.
- $f(x) = x^{1/3}$ shows that $f''(c)$ need not exist.
- If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.
- If c is a point of inflection, then the curve lies on different sides of the tangent line at c.
- If c is a point of inflection, then c is an isolated extremum of f'.

Point of inflection 2

- $f(x) = 2x^3 + x^3 \sin(1/x)$ below the tangent ($y = 0$) on one side and above the tangent on another, but $f'' = 12x + 6x \sin(1/x) - 4 \cos(1/x) - 1/x \sin(1/x)$ does not have fixed sign.
- $f(x) = x^3 + x^4 \sin(1/x)$,
  $f'(x) = 3x^2 - x^2 \cos(l/x) + 4x \sin(l/x)$. f' has an isolated minimum, but
  $f''(x) = 6x - \sin(l/x) - 6x \cos(l/x) + 12x^2 \sin(l/x)$ does not have fixed sign on either side of 0.
- $x^4 \sin(1/x)$ $f'(c) = f''(c) = 0$, but neither extremum nor point of inflection.

L'Hôpital's Rule

- Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,$$

  for $L \in \mathbb{R} \cup \infty$.

- Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,$$

  for $L \in \mathbb{R} \cup \infty$.

L'Hôpital's Rule 2

- L'Hôpital does not say that

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L \Longleftarrow \lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

- If $f(x) = x + \sin x$ and $g(x) = x$, then

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{1+\cos x}{1}$$

does not exist, while

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}(1+\frac{\sin x}{x}) = 1.$$