Numbers

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Why is $0.\overline{9} = 1$?

- Since $1/3 = 0.\overline{3}$, we can multiply by 3 and get

$$1 = 3 \cdot 1/3 = 3 \cdot 0.\overline{3} = 0.\overline{9}.$$

- We can also set $x = 0.\overline{9}$. Then $10x = 9.\overline{9}$, and if we subtract we get $9x = 9$ and finally $x = 1$.

Natural numbers

- Let $\mathbb{N}$ denote the positive, natural numbers $\{1, 2, 3, \ldots\}$.
- Remember that positive means > 0 and negative means < 0, so nonnegative is not the same as positive, but means positive or zero.

Why is $(-1)(-1) = 1$?

- One way to understand why $(-1)(-1) = 1$, is to say that multiplying by -1 is the same as "flipping" across zero on the number line, in which case, flipping twice does nothing. However, it can be instructive to also see an algebraic proof. Assume that we know how to multiply natural numbers, and that we want to extend this to integers. We want to do this in such a way that the following three properties are preserved.
  1. Commutative $ab = ba$
  2. Associative $(ab)c = a(bc)$
  3. Distributive $a(b + c) = ab + ac$

Why is $(-1)(-1) = 1$? 2

- Assume that $a, b \in \mathbb{N}$. We know that

$$a(-b) = (-b) + (-b) + \cdots + (-b) = -ab \qquad (1)$$

by repeated addition, and to compute $(-a)b$, we use Equation (1) and commutativity to get

$$(-a)b = b(-a) = -ba = -ab. \qquad (2)$$

Why is $(-1)(-1) = 1$? 3

- We want to show that $(-1)(-1) = 1$, and to do that, we consider $(-1)(-1) - 1$ and use distributivity

$$
\begin{aligned}
(-1)(-1) - 1 &= \\
(-1)(-1) + (-1) &= \\
(-1)(-1) + (-1) \cdot 1 &= \\
(-1)(-1 + 1) &= \\
(-1) \cdot 0 &= 0.
\end{aligned}
$$

Hence $(-1)(-1) = 1$.

Divison by zero

- The key to understanding division and fractions is that

$$\frac{a}{b} = c \iff a = bc. \tag{3}$$

This shows why we cannot divide by 0. If $b = 0$, we get

$$\frac{a}{0} = c \iff a = 0 \cdot c = 0,$$

which shows that we get a contradiction if we try to assign a value to $a/0$ when $a \neq 0$. But what if $a = 0$? In that case, the above equation just says that $0 = 0 \cdot c = 0$, which is true for any c. But that is precisely the problem. We could theoretically define $0/0$ to be anything, without violating (3), but which value should we choose? Since we theoretically could pick any value, we say that $0/0$ is an indeterminate form.

Divison by zero 2

- And even if we do not violate (3), could we get into other problems if we tried to pick a value for $0/0$? Consider the following equations

$$
\begin{aligned}
0+0&=0\\
2\cdot 0&=1\cdot 0\\
2&=1\cdot\frac{0}{0}.
\end{aligned}
$$

  This would give a contradiction if we tried to define $0/0$ to be anything other than 2, and if we tried to define it to be 2, we could just consider three zeros above, and get another contradiction. So we will simply say that $0/0$ is also undefined.

Divison by fractions

- Many students do not understand why dividing by a fraction is the same as inverting the second fraction and multiplying as follows

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b}\frac{d}{c}. \tag{4}$$

To see this, we must show that if multiply the number on the right by the divisor, we get the divident, i.e.,

$$\left(\frac{a}{b}\frac{d}{c}\right)\frac{c}{d} = \frac{a}{b}\left(\frac{d}{c}\frac{c}{d}\right) = \frac{a}{b}.$$

Divison by fractions 2

- Another way to see this, is to expand the fraction by the divisor as follows

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\frac{d}{c}}{\frac{c}{d}\frac{d}{c}} = \frac{\frac{a}{b}\frac{d}{c}}{1} = \frac{a}{b}\frac{d}{c}$$

or alternatively, to expand by the products of the denominators in the parts of the complex fraction as follows

$$\frac{\frac{a}{b}bd}{\frac{c}{d}bd} = \frac{ad}{bc}.$$

Powers

- Assume that we have defined a^n with $n \in \mathbb{N}$ to be

$$a^n = \overbrace{a \cdot \ldots \cdot a}^{n} \tag{5}$$

For $n, m \in \mathbb{N}$ it is easy to see that we have the following property

$$a^n a^m = \overbrace{a \cdots a}^{n}\overbrace{a \cdots a}^{m} = \overbrace{a \cdots a}^{n+m} = a^{n+m}. \tag{6}$$

We now want to extend Definition (5) to $n \in \mathbb{Z}$ in way that preserves property (6). In other words, we will assume that (6) holds, and see what that implies about a^0 and a^{-n} for $n \in \mathbb{N}$.

Powers 2

- Setting $m = 0$ in (6), we get

  $$a^n = a^{n+0} = a^n \cdot a^0,$$

  so if $a \neq 0$, we can divide by a^n and conclude that $a^0 = 1$. (We will discuss 0^0 later.) We now set $m = -n$ in (6) and get

  $$1 = a^0 = a^{n-n} = a^n a^{-n},$$

  so it follows that

  $$a^{-n} = \frac{1}{a^n}.$$

  We can now check that

  $$\frac{a^n}{a^m} = \frac{a^{n-m} a^m}{a^m} = a^{n-m}$$

  holds for $n, m \in \mathbb{Z}$.

Fractional Exponents

- Again we want to extend a definition to a larger set of numbers by preserving a property. We know that for $m, n \in \mathbb{N}$ we have

$$(a^n)^m = \underbrace{a^n \cdots a^n}_{m} = \underbrace{(\overbrace{a \cdots a}^{n}) \cdots (\overbrace{a \cdots a}^{n})}_{m} = \overbrace{a \cdots a}^{n \cdot m} = a^{n \cdot m} \quad (7)$$

  We want to extend the definition of a^n to $n \in \mathbb{Q}$, while maintaining property (7). To see how to do this, we simply write $x = a^{1/n}$. Then

$$x^n = (a^{1/n})^n = a^{\left(\frac{1}{n}n\right)} = a^1 = a$$

  so we see that

$$a^{1/n} = \sqrt[n]{a}.$$

  Using property (7) again, we get that

$$a^{m/n} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}.$$

Is $0^0 = 1$?

- We have seen that for $a \neq 0$ we have $a^0 = 1$, so $\lim_{a \to 0} a^0 = 1$. It therefore seems natural to define $a^0 = 1$. However, for $x > 0$, we have $0^x = 0$ and it follows that $\lim_{x \to 0^+} 0^x = 0$. This shows that the function $f(x, y) = x^y$ does not have a limit at $(0, 0)$ since we get different values depending on how we approach $(0, 0)$. It follows that f is not continuous at $(0, 0)$. That makes it harder to find a good value for 0^0, but not impossible. The floor function $f(x) = \lfloor x \rfloor$, which denotes the greatest integer less than or equal to x, is not continuous at $x = 0$, but that does not prevent us from saying that $\lfloor 0 \rfloor = 0$.

Is $0^0 = 1$? 2

- We often write a polynomial as

$$p(x) = \sum_{k=0}^{n} a_k x^n.$$

- However, then

$$p(0) = a_0 0^0 + \cdots + a_n 0^n = a_0 0^0,$$

and we are implicitly assuming that $0^0 = 1$.

Is $0^0 = 1$? 3

- We can also consider a power series like

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Then

$$f(0) = 1 = \sum_{n=0}^{\infty} 0^n = 0^0.$$

If we do not define 0^0 to be 1, we will have trouble with even simple expressions like this.

Is $0^0 = 1$? 4

- Another example is the Binominal Theorem

$$(a+b)^x = \sum_{k=0}^{\infty} \binom{x}{k} a^k b^{x-k}.$$

- Setting $a = 0$ on both sides and assuming $b \neq 0$ we get

$$b^x = (0+b)^x = \sum_{k=0}^{\infty} \binom{x}{k} 0^k b^{x-k} = \binom{x}{0} 0^0 b^x = 0^0 b^x,$$

  where, we have used that $0^k = 0$ for $k > 0$, and that $\binom{x}{0} = 1$.

- We see that we must set $0^0 = 1$ in order for the binomial theorem to be valid.

Is $0^0 = 1$? 5

- Another reason why $0^0 = 1$ is because x^0 is the “empty product”, which should 1. For the same reason $0! = 1$.
  In order for the differentiation rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

  to hold for $n = 1$ when $x = 0$, we also requires $0^0 = 1$.
  But we have to be careful with 0^0. We cannot compute with it like other numbers. We cannot say that

$$1 = 0^0 = 0^{n-n} = \frac{0^n}{0^n} = \frac{0}{0}.$$

  So to sum up, we can write $0^0 = 1$ to make expressions simple, but we cannot compute with it using for instance power rules.

Rational numbers

- We will study the rational numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

We want to show that $\sqrt{2}$ is irrational. We will need the following lemma

Lemma

A natural number a is even if and only if a^2 is even.

Rational numbers 2

Proof.

If a is even we can write $a = 2k$ with $k \in \mathbb{Z}$ and then $a^2 = (2k)^2 = 4k^2 = 2(2k^2)$ so we see that

$$a \text{ is even} \implies a^2 \text{ is even.}$$

In order to show the opposite, we will use that

$$p \implies q \quad \text{is the same as} \quad \neg p \impliedby \neg q.$$

So we will show that

$$a \text{ is odd} \implies a^2 \text{ is odd.}$$

If $a = 2k + 1$ with $k \in \mathbb{Z}$, then
$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ so a^2 is odd. □

Rational numbers 3

Theorem

$\sqrt{2}$ is irrational.

Proof.

We will assume that $\sqrt{2}$ is rational and can be written as a/b, where $a, b \in \mathbb{Z}$ are relatively prime, i.e., they have no common factors. Then

$$2 = \frac{a^2}{b^2} \quad \text{and} \quad 2b^2 = a^2,$$

and we see that a^2 is even. But then we know from the above Lemma that a is also even, so $a = 2k$ with $k \in \mathbb{Z}$ and

$$a^2 = (2k)^2 = 4k^2 = 2(b^2) \quad \text{or} \quad b^2 = 2k^2.$$

Since b^2 is even, it follows that b is also even. We have now shown that both a and b are even, but this contradicts the assumption that a and b are relatively prime. □