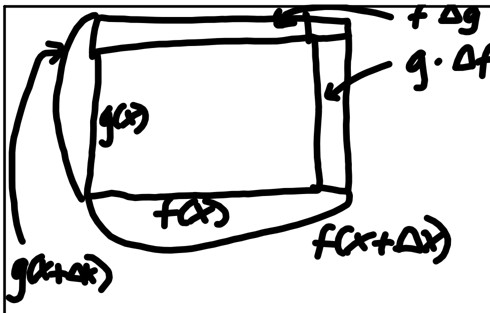


$$\Delta(fg) = f\Delta g + g\Delta f + \Delta f \Delta g$$

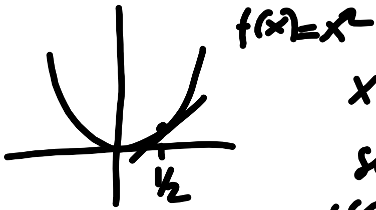
$$\frac{\Delta(fg)}{\Delta x} = f\frac{\Delta g}{\Delta x} + g\frac{\Delta f}{\Delta x} + \frac{\Delta f}{\Delta x}\Delta g$$

$$\downarrow \qquad\qquad \downarrow$$

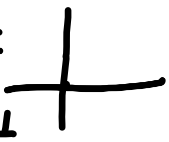
$$(fg)' \qquad fg' + gf' + f' \cdot 0$$

$$h(x) = f(g(x)) \quad h'(x) = f'(g(x))\, g'(x)$$

$$f(x) = x^2 \quad g(x) = \sin x \quad h(x) = (\sin x)^2$$



$x = \frac{\pi}{3}\ \frac{\pi}{6}$



$\sin\frac{\pi}{6} = \frac{1}{2}$

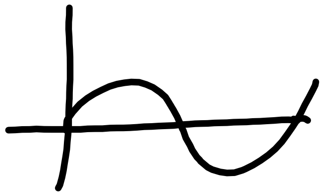
$f'(\frac{1}{2}) = 2(\frac{1}{2}) = 1$

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$y = g(x)$

$$= \lim_{\Delta x \to 0} \frac{\Delta f(g)}{\Delta g} \frac{\Delta g}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(y)}{\Delta y} \frac{\Delta g}{\Delta x} = f'(y)\, g'(x) = f'(g(x))\, g'(x)$$

vekst i $f(g)$ = (f vekst i g) $\cdot$ (vekst i g)



Kan $g(x) = g(x+\Delta x)$?

$\Delta x = \pi \quad x = 0$

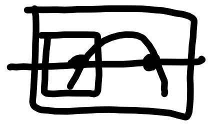
$\sin \pi = \sin 0$

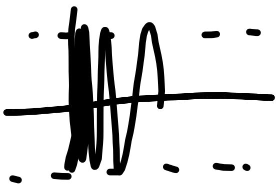
Hvis $\Delta x \approx 0$, så er $\sin(x+\Delta x) \neq \sin x$

Kan vi alltid anta at vi kan si

$$g(x+\Delta x) \neq g(x)$$

bare Δx er liten nok?



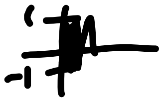


$\sin \frac{1}{x}$ er 0 uendelig mange ganger, vilkårlig nære $x = 0$

hvis $g(x)$ har isolerte nullpunkter, så OK.

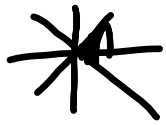
$$f_n(x)=\begin{cases} x^n \sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$$

$f_0(x)=\sin\frac{1}{x}$



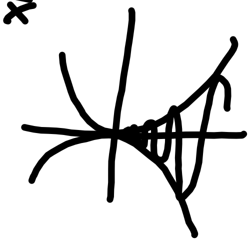
f_0 ikke kont i 0

$f_1(x)=x\sin\frac{1}{x}$

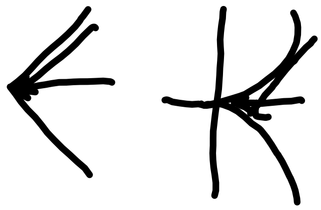


$\lim\limits_{x\to 0} f_1(x)=0$ f_1 er kont i 0.

$f_2(x)=x^2\sin\frac{1}{x}$



$f_2'(0)=0$



$$f_n(x)=\begin{cases} x^n \sin\frac{1}{x}, & x\neq 0\\ 0 & x=0\end{cases}$$

$\underset{x\neq 0}{f_1'(x)}=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$

$f_1'(0)=\lim\limits_{x\to 0}\frac{f_1(x)-f_1(0)}{x-0}=\lim\limits_{x\to 0}\frac{x\sin\frac{1}{x}}{x}=\lim\limits_{x\to 0}\sin\frac{1}{x}$ eks. ikke

$f_2'(0)=\lim\limits_{x\to 0}\frac{f_2(x)-f_2(0)}{x}=\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}=\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$

$\underset{x\neq 0}{f_2'(x)}=2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)=$

$2x\sin\frac{1}{x}-\cos\frac{1}{x}$

$f_2'(0)\neq\lim\limits_{x\to 0}f_2'(x)$

$\lim\limits_{x\to 0}f_2'(x)$ eks. ikke f_2' er ikke kont. i 0

f_2 er ikke <u>kontinuerlig deriverbar</u>

$\frac{\Delta y}{\Delta x}$ f_0 ikke kont

f_1 kont, ikke der

f_2, der, ikke kont der

f_3 ~~2 ganger der~~, f' f_3' kont, f_3'' eks ikke

$f_3(x) = x^3 \sin\frac{1}{x}$

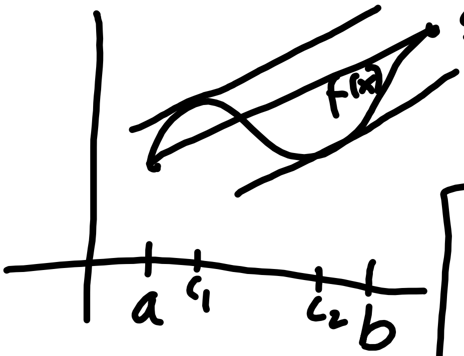
$f_3'(x) = 3x^2 \sin\frac{1}{x} + x^3 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) =$

$3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}$

$\lim_{x\to 0} f_3'(x) = 0 = f_3'(0)$ f_3' kont

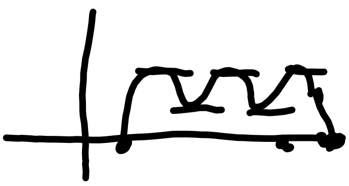
$f_3''(0) = \lim_{x\to 0} \frac{f_3'(x) - f_3'(0)}{x} = \lim_{x\to 0}\left(3x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ eks ikke!

f_4 f_4'' eks, f_4'' er ikke kont.



$\frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$

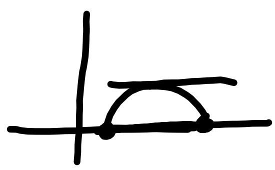
$f(a) = f(b) = 0$

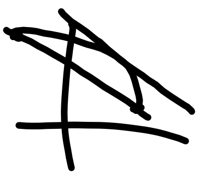


$g = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$

$h(x) = f(x) - g(x)$

$h(a) = h(b) = 0$





$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ voksende

$\Rightarrow f(x_1) < f(x_2)$ strengt voksende

$f(x) \ge 0 \Rightarrow \lim_{x \to a} f(x) \ge 0$

$f(x) > 0 \not\Rightarrow \lim_{x \to a} f(x) > 0$

$a_n = \frac{1}{n} > 0 \quad \lim_{n \to \infty} \frac{1}{n} = 0$

f er voksende på (a, b) (eller strengt voks.)

$c \in (a, b)$

$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \ge 0$

anta at $h > 0$

$h < 0$

$f(x) = x^3 \quad c = 0 \quad f'(0) = \lim_{h \to 0} \frac{h^3 - 0}{h} = \lim_{h \to 0} h^2 = 0$

Anta at $f' > 0$ på (a,b).

Velg $x_1 < x_2$ $\quad x_1, x_2 \in [a,b]$

Se på $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) > 0$

for en $c \in (x_1, x_2)$

$\Downarrow$

$f(x_1) < f(x_2)$

$\lim_{x\to\infty} \sin\frac{1}{x} = 0 \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{y\to 0} \frac{1}{y}\sin y = 1$

$y = \frac{1}{x} \quad x = \frac{1}{y}$

$x\to\infty \leftrightarrow y \to 0$

Du beveger deg langs $y = x^3$ med konstant fart.
Tangentvektoren måler din hastighet

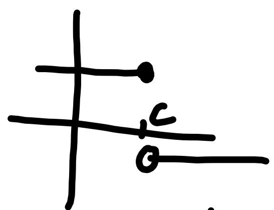
solver
hvarf – vende, snu
solstice, stand – stå
至 zhì ~~ekst~~ ekstrem

Er $f'(a)=\lim_{x\to a} f'(x)$? NOT!
Dette er def. av kont. deriverbar ⟺ f' er kont.

$x^2 + x^2\sin\frac{1}{x} = x^2(1+\sin\frac{1}{x})$

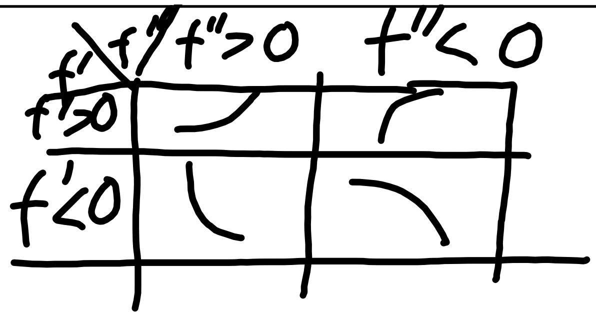
$2x^2 + x^2\sin\frac{1}{x} = x^2(2+\sin\frac{1}{x})$

Hvis $f'(c)>0$ og f' er kont, da i det

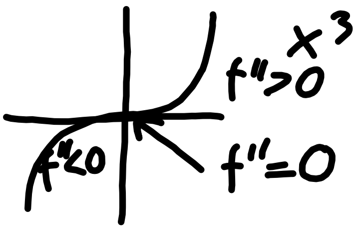


$f(c)>0$, f er ikke pos. på en interval rundt c.

f er kont og $f(c)>0 \Rightarrow f>0$ på et interval rundt c
$f(c)=\lim_{x\to c} f(x)$



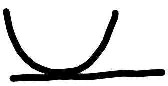
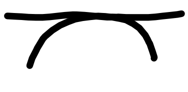
$f'' > 0$ krummer opp
$f'' < 0$ krummer ned



$x^3 + x$

c er et vendepunkt $\cancel{\overset{?}{\Rightarrow}}$ $f'(c) = 0$

$f'(c) = 0$ & $f''(c) > 0$ $\qquad$ $f'(c) = 0 \wedge f''(c) < 0$

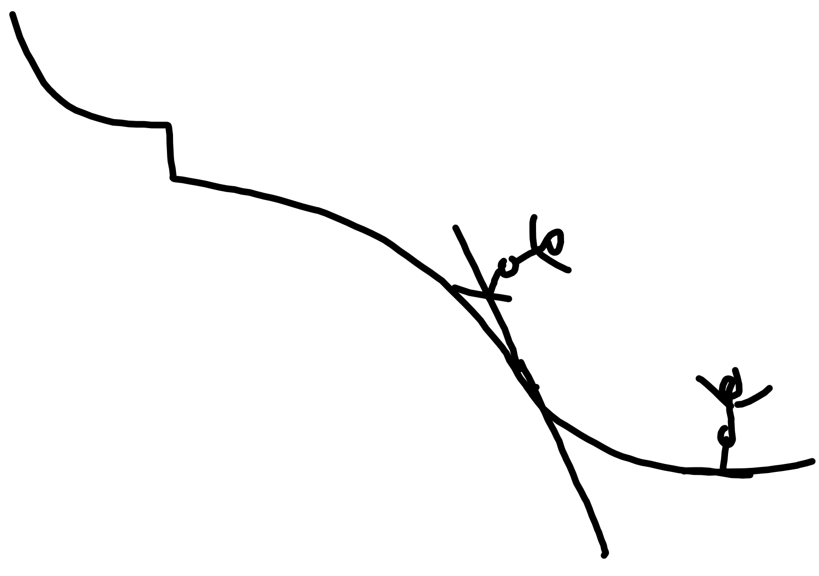
$f' = f'' = 0 \cancel{\overset{?}{\Rightarrow}}$ vendepunkt $f(x) = x^4$

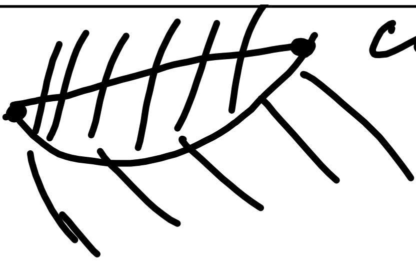
$f(x) = x^4$

$f'(x) = 4x^3$

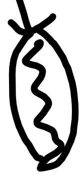
$f''(x) = 12x^2 \geq 0$

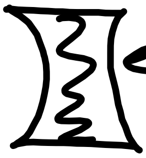
$f'(0) = f''(0) = 0$





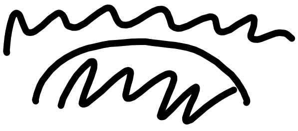
convex





concave

convex down = concave up



concave ~~up~~ down = convex ~~down~~ up

pl. of infl = change of concavity

ekstrem punkt $\neq$ monotonitet skifte

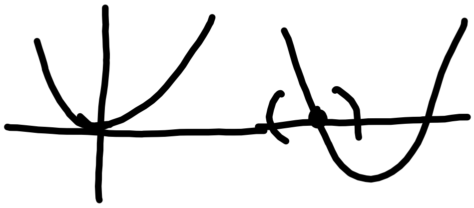
$\not\Rightarrow$ $2x^2 + x^2 \sin \frac{1}{x}$

vendepunkt = concavitet skifte

f'' skifter fortegn

f' skifter fortegn

moteksempler: x^4 x^3



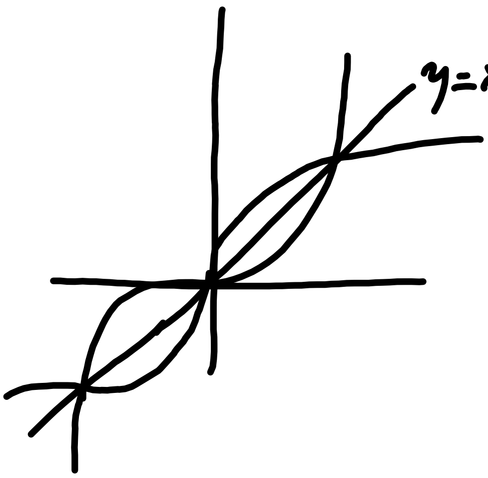
kysser krysser

kissing probler

→ skifter ikke fortegn

$x^2+x^2\sin\frac{1}{x}$

ikke-isolert nullpunkt



x^3 $\quad \sqrt[3]{x}=x^{1/3}$

$y=ax+b$

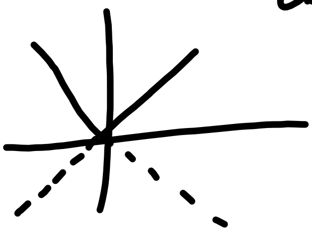
$ax+by+c=0$

alle linjer i planet

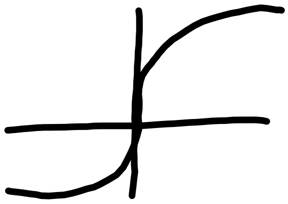
Hvis $b\neq 0$ $\quad y=-\frac{a}{b}x-\frac{c}{b}$

$b=0$ $\quad ax+c=0$ $\quad x=-\frac{c}{a}$

$f(x) = |x|$ ikke deriverbar fordi ikke tangent



$f(x) = x^{1/3}$



har tangent, men tangenten har ikke stigningstall

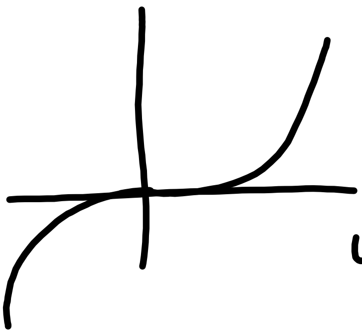
<u>ikke</u> deriverbar

vendepunkt, men $f'(0)$ og $f''(0)$ eks. <u>ikke</u>

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \qquad f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$

$f''(0)$ eks ikke



vendepunkt

Grenseverdier bevarer ikke strenge ulikheter. $a_n = \frac{1}{n} > 0$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$f(x) = x^3$ $f'(x) = 3x^2$

1) f strengt ~~d~~ voksende, men $f' \not> 0$

2) $f'(0) = 0$, men 0 er ikke et ekstremp. (terrassep.)

$f(x) = x^4$

$f''(0) = 0$ men ikke vendep.

vendep i c $\wedge$ $f''(c)$ eks $\Rightarrow f''(c) = 0$

Anta at $f''(x) > 0$ for $x > c$
og $f''(x) < 0$ for $x < c$

$$f''(c) = \lim_{x\to c} \frac{f'(x) - f'(c)}{x - c}$$

1) $x > c$. $\exists d$, $c < d < x$ slik at $\frac{f'(x) - f'(c)}{x - c} = f''(d) > 0$

$$\lim_{x\to c^+} \frac{f'(x) - f'(c)}{x - c} \geq 0$$

2) $x < c$ $\exists d$, $x < d < c$, s.a. $\frac{f'(x) - f'(c)}{x - c} = f''(d) < 0$

$$\lim_{x\to c^-} \frac{f'(x) - f'(c)}{x - c} \leq 0$$

$\Rightarrow f''(c) = 0$ siden $f''(c)$ eks.

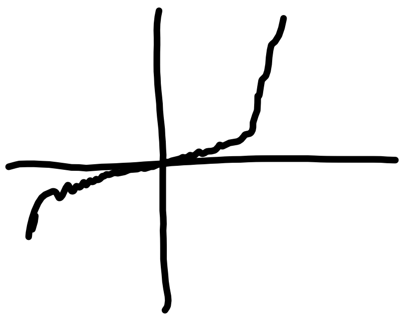
f har v. punkt → du vet noe om f'' rundt c.

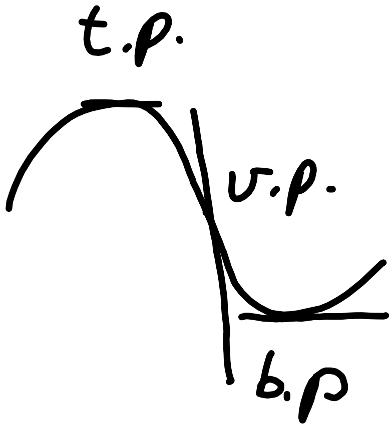
↓

du vet noe om f' rundt c via MVT (MVS)

↓

du vet noe om f'' i c (def. av der)





$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\frac{f(x)-f(0)}{x-0}}{\frac{g(x)-g(0)}{x-0}} = \lim_{x\to 0} \cancel{\frac{f'(c)}{g'(c)}} = L$$

~~Cauchys MVT~~ ~~c mellom x og 0.~~

Anta at f og g er kont. deriv.

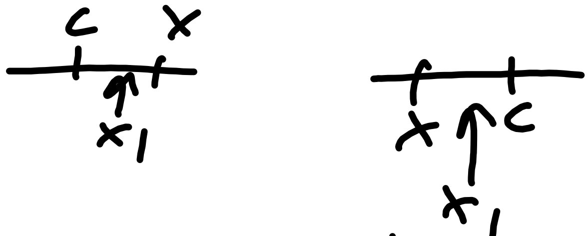
$$\frac{\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}}{\lim_{x\to 0} \frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

siden f og g er kont. d.

Tangentlinjen

$f(c)+f'(c)(x-c) \approx f(x)$ for x nær c

$f(c)+f'(x_1)(x-c) = f(x)$



$x-c$ og x_1-c har samme fortegn!

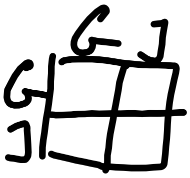
$\underline{x^n + x^k \sin\left(\frac{1}{x}\right)}$

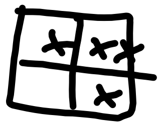
G,G
G,J
J,J

f'' skifter ikke fortegn fordi den skifter fortegn hele tiden!

$P(G,J) = \frac{1}{3}$, $P(G,G) = P(J,J) = \frac{1}{3\cancel{4}}$. NOT

$P(G,J) = \frac{1}{2}$, $P(J,J) = \frac{1}{4}$





$\{0,1\} = \{1,0\} = \cancel{\{0,0,1,1\}}$

$\binom{7}{5} = \binom{7}{2}$

uordnet ↔ like

med tilbakel. ↔ gjentagelse

$\binom{n}{k}$ 3 baller 5 farger

ulik farge

$\binom{5}{3} = \frac{5\cdot4\cdot3}{6} = 10$ $\binom{5+3-1}{3} = \binom{7}{3} = \frac{7\cdot6\cdot5}{6} = 35$

Kombinatorikk = antall mulige utvalg
↑
helt tall
Sannsynlighet = tall mellom 0 og 1

1. og 2. premie i matematikk → uten tilb ordnet

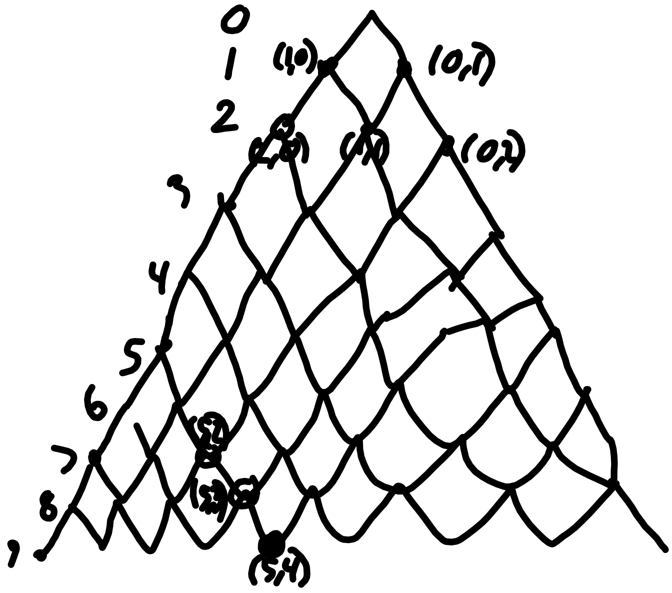
To like premier til de to beste i matematikk → uten tilb uordnet

Premie for flinkest i engelsk & ——"—— matematikk med tilbak.
Ulike premier ↦ ordnet Like premier ↦ uord.

(A) $1/2$
(B,A) $1/4$
(B,B) $1/4$

(A,A) $1/4$
(A,B) $1/4$
(B,A) $1/4$
(B,B) $1/4$

antall omganger



$r=1$, $s=2$

A: $\frac{1}{4}\left(\binom{2}{0}+\binom{2}{1}\right) = \frac{1}{4}(1+2) = \frac{3}{4}$

B: $\frac{1}{4}\binom{2}{2} = \frac{1}{4}$

$r=2$, $s=5$

A har ~~vert~~ vunnet 4
B — " — 1

(4,1)

1 (A,A,A,A,A,A)
6 (A,A,A,A,A,B), (A,A,A,A,B,A).....
$\binom{6}{2}=15$

6 (A,B,B,B,B,B), (B,A,B,B,B,B), ...
1 (B,B,B,B,B,B)

A: $\frac{2^6 - 7}{2^6}$

B: $\frac{7}{2^6}$

200 familier

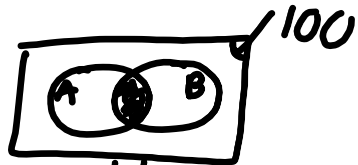
50 (G,G) ~~50~~ (G,J) ~~50~~ (J,G) ~~50 (J,J)~~
25 25

$P(G,G) = \frac{1}{2}$

$P(\{G,J\} \mid \text{Minst én gutt})$

$q = P(\text{Minst én gutt} \mid \{G,J\})$

Kjenner $P(A|B)$, vil finne $P(B|A)$



$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$P(B|A) \underset{\Uparrow}{=} P(B)$
A&B uavh

$$P(A\cap B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$\Rightarrow P(A|B)P(B)$

$|B| = 20$
$|A| = 30$
$|A\cap B| = 10$

$P(B|A) = \frac{10}{30} = \frac{1}{3}$
$P(B) = \frac{20}{100} = 0{,}2 \neq \frac{1}{3}$

$$\boxed{P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}}$$

$X \cup Y =$ utfallsrommet
$X \cap Y = \emptyset$

$$P(A) = P(A\cap X) + P(A\cap Y) =$$
$$P(A|X)P(X) + P(A|Y)P(Y)$$

Får barn til man har fått en sønn.
Stopper etter 4.

G	1/2	= 8/16
J, G	1/4	= 4/16
J, J, G	1/8	= 2/16
J, J, J, G	1/16	
J, J, J, J	1/16	
	1	

$$G: 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + 1\cdot\frac{1}{16} = \frac{15}{16}$$

$$J: 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + 3\cdot\frac{1}{16} + 4\cdot\frac{1}{16} = \frac{4+4+3+4}{16} = \frac{15}{16}$$

Uniform?
Valg?

1) 3 kort

S/H | S/H

~~S/S~~ H/H S/H

Du trekker H. Er det H på den andre siden også?

a) H/H og S/H like sanns. ⇒ $P = 1/2$

b) 2/3 for at kortet har to like sider ⇒ $P = 2/3$

b2) 1: S_1/S_2 2: H_1/H_2 3: S/H

$\cancel{(1, S_1)}, \cancel{(1, S_2)}, (2, H_1), (2, H_2), \cancel{(3, S)}, (3, H)$

2) Principle of restricted choice

Øst har kastet en konge. Har Øst K, D eller bare K?

Ø = K

$$\frac{Ø = K,D \text{ og } Ø \text{ spiller } K}{Ø = K,D \text{ og spiller } D} = Ø = K,D$$

P(Ø = K,D | Ø spilte K) = 1/3

P(Ø = K,D) = 1/2

3) 3 fanger

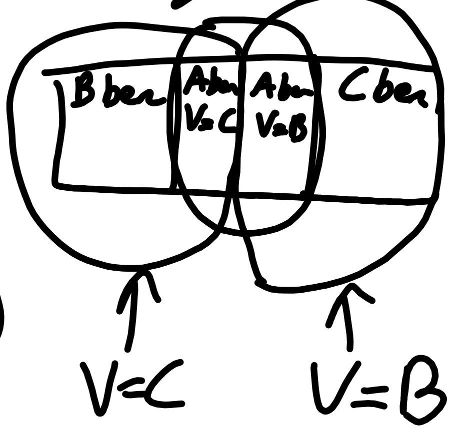
Vokter forteller A at B skal henges.

A ben.   C ben.

P = 1/2

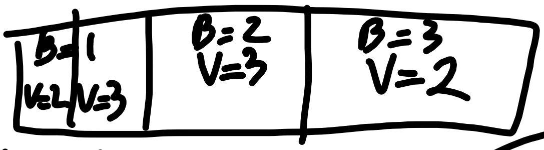
$$\frac{(A \text{ ben} \mid \text{Vokter sier } B)}{(A \text{ ben} \mid \text{Vokter sier } C)}$$

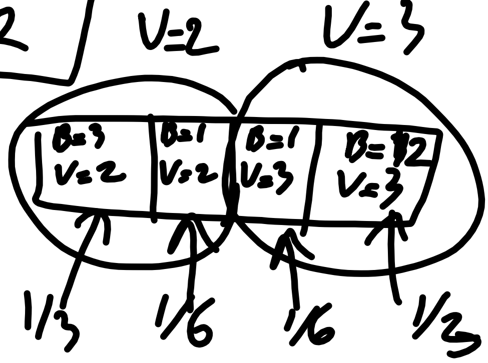
P(A ben | Vokter sier B) = $\frac{1}{3}$



4) Monty Hall

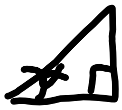
3 fanger   3 dører
guvernør   deltaker
vokter   vert



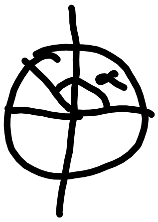


Bilen bak dør i.
B=i
Spiller velger dør 1

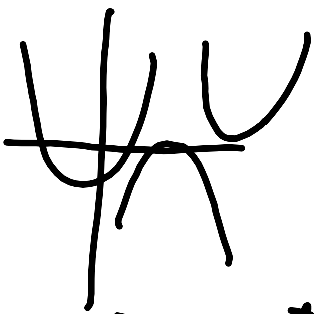
jiva
jiba   jb
jaib – bukt
komplementær vinkel $90-x$
$\cos x = \sin(90-x)$
cosinus er sinus til komplementær v.

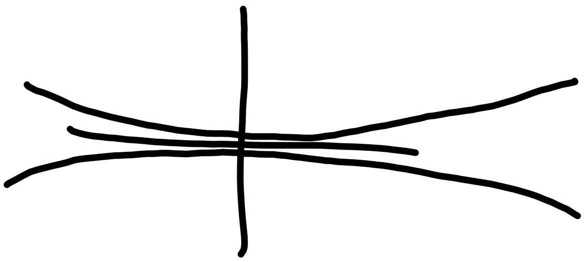


$0<\alpha<90$



$0\le\alpha\le 180$





p(x) er et polynom

$p(a)=0 \Leftrightarrow (x-a)\mid p(x)$

$p(x)=(x-a)\,q(x)$

$p(a)=(a-a)\,q(a)=0$

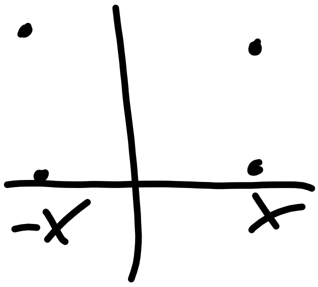
$p(x)=a(x-x_1)(x-x_2)$

For sym om y-aksen

$$\Downarrow$$

$$f(-x) = f(x)$$

$$f(x) = ax^2 + bx + c$$



---

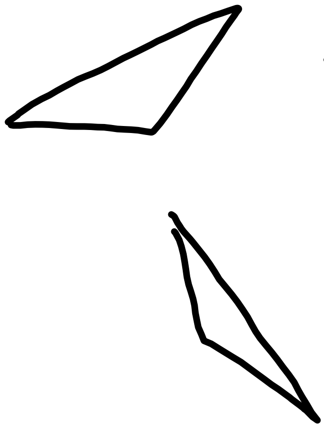
a $\quad ax^2 + bx + c$

$$a(x-x_1)(x-x_2) =$$

$$ax^2 - a(x_1+x_2)x + ax_1x_2$$

a og b har forskj. fortegn $\Leftrightarrow$

$$x_1 + x_2 > 0$$



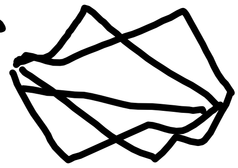
3 sider

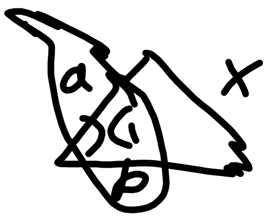
3 vinkler

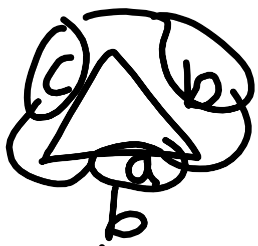
rettvinklet △

[illegible]

høyder 3

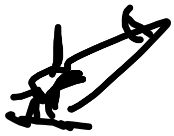






$$x^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = x^2 + b^2 - 2b\cos C\,x$$



vinkelen mellom horisonten
og solens bane er
90 − breddegrad

The tropics is the area between
the Tropics.

Steradianer − vinkelmål i $\mathbb{R}^3$