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Decimal Expansion of Rational Numbers

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Why is $0.999\ldots = 1$?

- We will write $1/3 = 0.333\ldots$ as $0.\overline{3}$ and call 3 the repetend.
- We can multiply by 3 and get

$$1 = 3 \cdot 1/3 = 3 \cdot 0.\overline{3} = 0.\overline{9}.$$

- We can also write

$$\begin{aligned} x &= 0.\overline{9} \\ 10x &= 9.\overline{9} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

Why is $0.999\ldots = 1$? 2

- Since $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ for $|x| < 1$, we have

$$0.\overline{9} = 9(0.1 + 0.01 + 0.001 + \cdots) = 9(0.1 + 0.1^2 + 0.1^3 + \cdots)$$
$$= 9\sum_{k=1}^{\infty} 0.1^k = 9 \cdot 0.1 \sum_{k=0}^{\infty} 0.1^k = 0.9\frac{1}{1-0.1} = \frac{0.9}{0.9} = 1.$$

Why is $0.999\ldots = 1$? 3

- Finally, we can argue that they have to be equal, since if they were not equal, we could find some number between them. However, there is no way to put any number between them.
- In general, we claim that

$$a.a_1a_2\ldots a_n = a.a_1a_2\ldots(a_n - 1)\overline{9}, \tag{1}$$

where $a_i \in \{0, \ldots, 9\}$, $a_n \neq 0$ and $a \in \mathbb{Z}$. For instance

$$3.14 = 3.13\overline{9} \quad \text{and} \quad -3.14 = -3.13\overline{9}.$$

Why is $0.999\ldots = 1$? 4

- $$\begin{aligned} & a.a_1a_2\ldots(a_n-1)\overline{9} \\ &= a.a_1a_2\ldots(a_n-1) + 9\sum_{k=n+1}^{\infty} 0.1^k \\ &= a.a_1a_2\ldots(a_n-1) + 9\cdot 0.1^{n+1}\sum_{k=0}^{\infty} 0.1^k \\ &= a.a_1a_2\ldots(a_n-1) + 9\cdot 0.1^{n+1}\frac{1}{1-0.1} \\ &= a.a_1a_2\ldots(a_n-1) + 0.1^{n+1}\frac{9}{0.9} \\ &= a.a_1a_2\ldots(a_n-1) + 0.1^{n+1}\cdot 10 \\ &= a.a_1a_2\ldots(a_n-1) + 0.1^{n} \\ &= a.a_1a_2\ldots(a_n-1) + 0.\overbrace{0\ldots0}^{n-1}1 \\ &= a.a_1a_2\ldots(a_n-1+1) = a.a_1a_2\ldots a_n. \end{aligned}$$

Why is $0.999\ldots = 1$? 5

- We see that every finite decimal expansion can also be written as an infinite decimal expansion. There is only exception, namely 0.
- One way to understand why 0 is exceptional is because for positive numbers, the infinite expansion “looks” smaller, wile for negative numbers, the infinite expansion “looks” bigger. So it is not surprising that 0 is a singular case.

Decimal Expansion

- **Theorem**

  A number is rational if and only if the decimal expansion is finite or repeating.

  - Proof: We will for simplicty assume that $0 < x < 1$. $\Longleftarrow$: Assume that x has a finite decimal expansion. Then

    $$x = 0.a_1 a_2 ... a_n = \frac{a_1 a_2 ... a_n}{10^n},$$

    which is a fraction of integers.

Decimal Expansion 2

- Assume that x has a periodic decimal expansion. We can find a number s so that the decimal expansion of $10^s x$ starts repeating right after the decimal point. Then

  $$\begin{aligned} 10^s x &= a.\overline{a_1 ... a_r} \\ (10^r - 1)10^s x &= (10^r - 1)a + a_1 \ldots a_r \\ x &= \frac{(10^r - 1)a + a_1 \ldots a_r}{(10^r - 1)10^s} \end{aligned}$$

  which is rational.

Decimal Expansion 3

- Alternatively, we can use infinite series, and write

$$
\begin{aligned}
10^s x = a.\overline{a_1 \dots a_r} &= a + \sum_{k=1}^{\infty} a_1 \dots a_r 10^{-rk} \\
&= a + \frac{a_1 \dots a_r}{10^r} \sum_{k=0}^{\infty} 10^{-rk} \\
&= a + \frac{a_1 \dots a_r}{10^r} \left(\frac{1}{1 - 10^{-r}} \right) \\
&= \frac{(10^r - 1)a + a_1 \dots a_r}{10^r - 1},
\end{aligned}
$$

  which is rational.
- $\Longrightarrow$: If $x = \frac{m}{n}$ the division will either terminate, or we will get repeating remainders after at most $n - 1$ steps.

Decimal Expansion 4

- As an example, consider $1/7$.

$$
\begin{array}{ll}
1 : 7 = 0.142857\dots & \\
-0 & \\
\overline{10} & \text{remainder } 1 \\
-7 & \\
\overline{30} & \text{remainder } 3 \\
-28 & \\
\overline{20} & \text{remainder } 2 \\
-14 & \\
\overline{60} & \text{remainder } 6 \\
-56 & \\
\overline{40} & \text{remainder } 4 \\
-35 & \\
\overline{50} & \text{remainder } 5 \\
-49 & \\
\overline{1} & \text{remainder } 1
\end{array}
$$

Decimal Expansion 5

- Euler's Theorem says that if $(n, 10) = 1$, then $10^{\phi(n)} \equiv 1 \pmod{n}$, which is the same as saying that $n | (10^{\phi(n)} - 1)$, i.e., n divides a "9-block" of length $\phi(n)$.
- From the decimal expansion of $1/7$, we see that

$$(10^6 - 1)1/7 = 142857.142857\ldots - 0.142857 = 142857,$$

so that $999999 = 7 \cdot 142857$. This shows that 7 divides a "9-block", $10^6 - 1$, of length equal to the period of $1/7$ and that $(10^6 - 1)/7$ is the repetend. Since $\phi(7) = 6$, this agrees with Euler's Theorem.

Types of Rational Decimal Expansion

- Consider $\frac{m}{n}$ where $0 < m < n$ and $(m, n) = 1$.

Terminating	$0.d_1 \ldots d_t$	$\frac{m}{2^u 5^v}$, $t = \max(u, v)$	$\frac{M_t}{10^t} = \frac{d_1 \ldots d_t}{10^t}$
Simple-periodic	$0.\overline{d_1 \ldots d_r}$	$\frac{m}{n}$, $(n, 10) = 1$	$\frac{M_s}{10^r - 1} = \frac{d_1 \ldots d_r}{10^r - 1}$
Delayed-periodic	$0.d_1 \ldots d_t\overline{d_{t+1} \ldots d_{t+r}}$	$\frac{m}{n_1 n_2}$, $n_1 = 2^u 5^v$, $(n_2, 10) = 1$, $t = \max(u, v) > 1$, $n_2 > 1$.	$\frac{M_d}{10^t(10^r - 1)}$

- Since $M_d < 10^t(10^r - 1)$, we can divide by $(10^r - 1)$ to get a quotient with at most t digits.

$$M_d = (10^r - 1)d_1 \ldots d_t + d_{t+1} \ldots d_{t+r} = 10^r d_1 \ldots d_t + d_{t+1} \ldots d_{t+r} - d_1 \ldots d_t$$
$$= d_1 \ldots d_{t+r} - d_1 \ldots d_t, \quad \text{and}$$
$$\frac{M_d}{10^t(10^r - 1)} = \frac{d_1 \ldots d_t}{10^t} + \frac{d_{t+1} \ldots d_{t+r}}{10^t(10^r - 1)},$$

which shows how to convert between M_d and the d_i.

Types of Rational Decimal Expansion 2

- In the finite case, there might be initial zeros in $d_1 \dots d_t$, but there are no terminal zeros, since we assume that d_t is the last nonzero digit. That means that $d_1 \dots d_t$ id not divisible by 10, so we can cancel some 2s or some 5s, but not both. We therefore have $t = \max(u, v)$.
- In the delayed-periodic case, we have $M_d = d_1 \dots d_{t+r} - d_1 \dots d_t$. In order for M_d to be divisible by 10, we must have $d_t = d_{t+r}$, but in that case we could instead make d_t part of the repend, i.e., $0.d_1 \dots d_{t-1}\overline{d_t \dots d_{t+r-1}}$.

Types of Rational Decimal Expansion 3

- Proof: m/n is terminating if and only if

$$m/n = \frac{m}{2^u 5^v} = \frac{M_t}{10^t}.$$

- m/n is simple-periodic if and only if we can cancel the decimals by shifting one period, i.e.

$$(10^r - 1)m/n = M_s.$$

- m/n is delayed-periodic if and only if we can cancel the decimals by shifting one period and moving the period t places, i.e

$$10^t(10^r - 1)m/n = M_d. \quad \square$$

- Notice that there may be initial 0's in the d_i's.
-

$$0.\overline{062} = 62/999, \quad 0.0\overline{62} = 62/(10 \cdot 99) = 62/990.$$

- Notice that the fractions in the last column need not be reduced.

Types of Rational Decimal Expansion 4

- In the simple-periodic case, the repetend is simply $m(10^r - 1)/n$, but in the delayed-periodic case, we must divide $m10^t(10^r - 1)/n$ by $10^r - 1$ to separate the finite and repeating parts. However, it is easier to divide $m10^t/n_1$ by n_2 to keep the numbers smaller, as the following examples show.
- $$\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{5}{10\cdot 3} = \frac{1\cdot 3+2}{10\cdot 3} = \frac{1}{10} + \frac{2}{10\cdot 3} = 0.1\overline{6},$$
  $$\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{5\cdot 3}{10\cdot 9} = \frac{15}{10\cdot 9} = \frac{1\cdot 9+6}{10\cdot 9} = \frac{1}{10} + \frac{6}{10\cdot 9} = 0.1\overline{6}.$$
- $$\frac{1}{24} = \frac{1}{2^3\cdot 3} = \frac{5^3}{10^3\cdot 3} = \frac{125}{10^3\cdot 3} = \frac{41\cdot 3+2}{10^3\cdot 3} = \frac{41}{10^3} + \frac{2}{10^3\cdot 3} = 0.041\overline{6},$$
  $$\frac{1}{24} = \frac{1}{2^3\cdot 3} = \frac{5^3\cdot 3}{10^3\cdot 9} = \frac{375}{10^3\cdot 9} = \frac{41\cdot 9+6}{10^3\cdot 9} = \frac{41}{10^3} + \frac{6}{10^3\cdot 9} = 0.041\overline{6}.$$

Types of Rational Decimal Expansion 5

- Notice that $10^6 - 1 = 3^3\cdot 7\cdot 11\cdot 13\cdot 37 = 76923\cdot 13 = 142857\cdot 7$.
  $$\frac{1}{26} = \frac{1}{2\cdot 13} = \frac{5\cdot 76923}{10\cdot(10^6-1)} = \frac{384615}{10\cdot(10^6-1)} = 0.0\overline{384615}.$$
- $$\frac{1}{28} = \frac{1}{2^2\cdot 7} = \frac{25}{10^2\cdot 7} = \frac{3\cdot 7+4}{10^2\cdot 7} = \frac{3}{10^2} + \frac{4}{10^2\cdot 7} = 0.03\overline{571428},$$
  $$\frac{1}{28} = \frac{1}{2^2\cdot 7} = \frac{25\cdot 142857}{10^2\cdot(10^6-1)} = \frac{3\cdot(10^6-1)+571428}{10^2\cdot(10^6-1)}$$
  $$= \frac{3}{10^2} + \frac{571428}{10^2\cdot(10^6-1)} = 0.03\overline{571428}.$$
- Notice how the type of the decimal expansion of m/n and the size of r and t only depends on n.

Cyclic Numbers (Optional)

- Consider the following decimal expansions

$$\begin{aligned} 1/7 &= 0.\overline{142857} \\ 2/7 &= 0.\overline{285714} \\ 3/7 &= 0.\overline{428571} \\ 4/7 &= 0.\overline{571428} \\ 5/7 &= 0.\overline{714285} \\ 6/7 &= 0.\overline{857142} \end{aligned}$$

- Notice how the digits of the repetends are cyclic permutations of each other, and that they are obtained by multiplying 142857.

Multiple Cycles (Optional)

- Sometimes the numbers m/n break into several cycles. For example, the multiples of 1/13 can be divided into two sets:

$$\begin{aligned} 1/13 &= 0.\overline{076923} \\ 10/13 &= 0.\overline{769230} \\ 9/13 &= 0.\overline{692307} \\ 12/13 &= 0.\overline{923076} \\ 3/13 &= 0.\overline{230769} \\ 4/13 &= 0.\overline{307692} \end{aligned}$$

  where each repetend is a cyclic re-arrangement of 076923 and

$$\begin{aligned} 2/13 &= 0.\overline{153846} \\ 7/13 &= 0.\overline{538461} \\ 5/13 &= 0.\overline{384615} \\ 11/13 &= 0.\overline{846153} \\ 6/13 &= 0.\overline{461538} \\ 8/13 &= 0.\overline{615384} \end{aligned}$$

  where each repetend is a cyclic re-arrangement of 153846.

- The first set corresponds to remainders of $1, 3, 4, 9, 10, 12$, while the second set corresponds to remainders of $2, 5, 6, 7, 8, 11$.

Period of Periodic Decimals

- We see from
$$(10^r - 1)m = Mn$$
that there is a repeating block of length r if and only if $10^r \equiv 1 \pmod{n}$.
- This block could itself consist of repeating parts, but if define the period of a periodic decimal to be the length of the minimal repeating block, i.e. the repetend, then the period is equal to the order of 10 mod n.
- We know from Euler's Theorem that if $(n, 10)$, then n divides $10^{\phi(n)} - 1$, so the period divides $\phi(n)$.

Factoring $10^n - 1$

- To find denominators with short periods, we use the following table. The period of $1/p$ is the r for which p first appears as a factor in $10^r - 1$. Notice how 3, 11 and 13 appear earlier that given by Euler's Theorem, while 7 first appears in $10^6 - 1$.

$$10^1 - 1 = 3^2$$
$$10^2 - 1 = 3^2 \cdot 11$$
$$10^3 - 1 = 3^3 \cdot 37$$
$$10^4 - 1 = 3^2 \cdot 11 \cdot 101$$
$$10^5 - 1 = 3^2 \cdot 41 \cdot 271$$
$$10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$$
$$10^7 - 1 = 3^2 \cdot 239 \cdot 4649$$
$$10^8 - 1 = 3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137$$
$$10^9 - 1 = 3^4 \cdot 37 \cdot 333667$$
$$10^{10} - 1 = 3^2 \cdot 11 \cdot 41 \cdot 271 \cdot 9091$$
$$10^{11} - 1 = 3^2 \cdot 21649 \cdot 513239$$
$$10^{12} - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901$$

Summary of $1/n$

- t is the length of the terminating part and r is the length of the repetend in the decimal expansion of $1/n$.

$1/n$	t	r	$\phi(n)$	$1/n$	t	r	$\phi(n)$
$1/2 = 0.5$	1			$1/22 = 0.0\overline{45}$	1	2	10
$1/3 = 0.\overline{3}$		1	2	$1/23 = 0.\overline{0434782608695652173913}$		22	22
$1/4 = 0.25$	2			$1/24 = 0.041\overline{6}$	3	1	8
$1/5 = 0.2$	1			$1/25 = 0.04$	2		
$1/6 = 0.1\overline{6}$	1	1	2	$1/26 = 0.0\overline{384615}$	1	6	12
$1/7 = 0.\overline{142857}$		6	6	$1/27 = 0.\overline{037}$		3	18
$1/8 = 0.125$	3			$1/28 = 0.03\overline{571428}$	2	6	12
$1/9 = 0.\overline{1}$		1	6	$1/29 = 0.\overline{0344827586206896551724137931}$		28	28
$1/10 = 0.1$	1			$1/30 = 0.0\overline{3}$	1	1	8
$1/11 = 0.\overline{09}$		2	10	$1/31 = 0.\overline{032258064516129}$		15	30
$1/12 = 0.08\overline{3}$	2	1	4	$1/32 = 0.03125$	5		
$1/13 = 0.\overline{076923}$		6	12	$1/33 = 0.\overline{03}$		2	20
$1/14 = 0.0\overline{714285}$	1	6	6	$1/34 = 0.0\overline{2941176470588235}$	1	16	16
$1/15 = 0.0\overline{6}$	1	1	8	$1/35 = 0.0\overline{285714}$	1	6	24
$1/16 = 0.0625$	4			$1/36 = 0.02\overline{7}$	2	1	12
$1/17 = 0.\overline{0588235294117647}$		16	16	$1/37 = 0.\overline{027}$		3	36
$1/18 = 0.0\overline{5}$	1	1	6	$1/38 = 0.0\overline{263157894736842105}$	1	18	18
$1/19 = 0.\overline{052631578947368421}$		18	18	$1/39 = 0.\overline{025641}$		6	24
$1/20 = 0.05$	2			$1/40 = 0.025$	3		
$1/21 = 0.\overline{047619}$		6	12	$1/41 = 0.\overline{02439}$		5	40

- What can you say about $1/27$ and $1/37$? Why?
- $10^3 - 1 = 3^3 \cdot 37$.

Primes with Given Period

- Primes p with repeating decimal expansions of period r in $1/p$.

Period	Primes
1	3
2	11
3	37
4	101
5	41, 271
6	7, 13
7	239, 4649
8	73, 137
9	333667
10	9091
11	21649, 513239
12	9901
13	53, 79, 265371653
14	909091
15	31, 2906161
16	17, 5882353
17	2071723, 5363222357
18	19, 52579
19	1111111111111111111
20	3541, 27961

- Notice how 7, 17 and 19 have maximal periods, $p - 1$. Gauss conjectured in 1801 that there are infinitely many primes with maximal periods, but this has not been proved.

Periods of Inverse Primes

- Here are the periods of $1/p$ for all primes less than 101 except for 2 and 5.

p	r	p	r	p	r
3	1	31	15	67	33
7	6	37	3	71	35
11	2	41	5	73	8
13	6	43	21	79	13
17	16	47	46	83	41
19	18	53	13	89	44
23	22	59	58	97	96
29	28	61	60	101	4

Periods of $1/n$ (Optional)

- If $n = p^k$, the period is a divisor of $\phi(p^k) = (p-1)p^{k-1}$, but there is no simple formula.
- If $n = n_1 n_2$ where $(n_1, n_2) = 1$, then it can be shown that the period of $1/n$ is the least common multiple of the periods of $1/n_1$ and $1/n_2$.