Numbers

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Natural numbers

- Let $\mathbb{N}$ denote the positive, natural numbers $\{1, 2, 3, \ldots\}$.
- Remember that positive means > 0 and negative means < 0, so nonnegative is not the same as positive, but means positive or zero.

Why is $(-1)(-1) = 1$?

- One way to understand why $(-1)(-1) = 1$, is to say that multiplying by -1 is the same as "flipping" across zero on the number line, in which case, flipping twice does nothing.
- However, it is instructive to also see an algebraic proof. Assume that we know how to multiply natural numbers, and that we want to extend this to integers. We want to do this in such a way that the following three properties are preserved.
  1. Commutative $ab = ba$
  2. Associative $(ab)c = a(bc)$
  3. Distributive $a(b + c) = ab + ac$

Why is $(-1)(-1) = 1$? 2

- Assume that $a, b \in \mathbb{N}$. We know that

$$a(-b) = (-b) + (-b) + \cdots + (-b) = -ab \tag{1}$$

  by repeated addition.
- To compute $(-a)b$, we use commutativity and Equation (1) to get

$$(-a)b = b(-a) = -ba = -ab.$$

Why is $(-1)(-1) = 1$? 3

- We want to show that $(-1)(-1) = 1$, and to do that, we consider $(-1)(-1) - 1$ and use distributivity

$$
\begin{aligned}
(-1)(-1) - 1 &= \\
(-1)(-1) + (-1) &= \\
(-1)(-1) + (-1) \cdot 1 &= \\
(-1)(-1 + 1) &= \\
(-1) \cdot 0 &= 0.
\end{aligned}
$$

Hence $(-1)(-1) = 1$.

Division by zero

- The key to understanding division and fractions is that

$$\frac{a}{b} = c \iff a = bc. \tag{2}$$

This shows why we cannot divide by 0. If $b = 0$, we get

$$\frac{a}{0} = c \iff a = 0 \cdot c = 0,$$

which shows that we get a contradiction if we try to assign a value to $a/0$ when $a \neq 0$.

- But what if $a = 0$? In that case, the above equation just says that $0 = 0 \cdot c = 0$, which is true for any c. But that is precisely the problem. We could theoretically define $0/0$ to be anything, without violating (2), but which value should we choose? Since we theoretically could pick any value, we say that $0/0$ is an indeterminate form.

Division by fractions

- Many students do not understand why dividing by a fraction is the same as inverting the second fraction and multiplying

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b}\frac{d}{c}. \tag{3}$$

To see this, we must show that if multiply the number on the right by the divisor, we get the dividend, i.e.,

$$\left(\frac{a}{b}\frac{d}{c}\right)\frac{c}{d} = \frac{a}{b}\left(\frac{d}{c}\frac{c}{d}\right) = \frac{a}{b}.$$

Division by fractions 2

- Another way to see this is to use complex fractions

$$\frac{\frac{a}{b}bd}{\frac{c}{d}bd} = \frac{ad}{bc}.$$

Division by fractions 3

- It is also instructive to consider unit fractions, $1/d$. There are many ways to argue that

$$a : \frac{1}{d} = ad.$$

- It then follows from associativity that

$$\left(\frac{a}{b}\right) : \frac{c}{d} = \left(\left(\frac{1}{b}a\right) : \left(\frac{1}{d}\right)\right) : c = \frac{1}{b}(ad) : c = \frac{ad}{bc}.$$

Powers

- Assume that we have defined a^n with $n \in \mathbb{N}$ to be

$$a^n = \overbrace{a \cdot \ldots \cdot a}^{n}. \tag{4}$$

For $n, m \in \mathbb{N}$ it is easy to see that we have the following property

$$a^n a^m = \overbrace{a \cdots a}^{n}\overbrace{a \cdots a}^{m} = \overbrace{a \cdots a}^{n+m} = a^{n+m}. \tag{5}$$

- We now want to extend Definition (4) to $n \in \mathbb{Z}$ in way that preserves property (5). In other words, we will assume that (5) holds, and see what that implies about a^0 and a^{-n} for $n \in \mathbb{N}$.

Powers 2

- Setting $m = 0$ in (5), we get

$$a^n = a^{n+0} = a^n \cdot a^0,$$

so if $a \neq 0$, we can divide by a^n and conclude that $a^0 = 1$. (We will discuss 0^0 later.)

- We now set $m = -n$ in (5) and get

$$1 = a^0 = a^{n-n} = a^n a^{-n},$$

so it follows that

$$a^{-n} = \frac{1}{a^n}.$$

- We can now check that

$$\frac{a^n}{a^m} = \frac{a^{n-m}a^m}{a^m} = a^{n-m}$$

holds for $n, m \in \mathbb{Z}$.

Powers 3

- Another way to understand this, is to interpret a^0 as an "empty product". The "empty sum" $0 \cdot a$ is the additive identity 0, while the "empty product" a^0 is the multiplicative identity 1.

Fractional Exponents

- Again we want to extend a definition to a larger set of numbers by preserving a property. We know that for $m, n \in \mathbb{N}$ we have

$$(a^n)^m = \underbrace{a^n \cdots a^n}_{m} = \underbrace{(\overbrace{a \cdots a}^{n}) \cdots (\overbrace{a \cdots a}^{n})}_{m} = \overbrace{a \cdots a}^{n \cdot m} = a^{n \cdot m}. \quad (6)$$

  We want to extend the definition of a^n to $n \in \mathbb{Q}$, while maintaining property (6). To see how to do this, we simply write $x = a^{1/n}$.
- Then

$$x^n = (a^{1/n})^n = a^{\left(\frac{1}{n}n\right)} = a^1 = a$$

  so we see that

$$a^{1/n} = \sqrt[n]{a}.$$

- Using property (6) again, we get that

$$a^{m/n} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}.$$

Is $0^0 = 1$?

- We have seen that for $a \neq 0$ we have $a^0 = 1$, so $\lim_{a \to 0} a^0 = 1$. It therefore seems natural to define $0^0 = 1$. However, for $x > 0$, we have $0^x = 0$ and it follows that $\lim_{x \to 0^+} 0^x = 0$.
- This shows that the function $f(a, x) = a^x$ does not have a limit at $(0, 0)$ since we get different values depending on how we approach $(0, 0)$. It follows that f is not continuous at $(0, 0)$.
- That makes it harder to find a good value for 0^0, but not impossible.

Is $0^0 = 1$? 2

- We often write a polynomial as

$$p(x) = \sum_{k=0}^{n} a_k x^k.$$

- However, then

$$p(0) = a_0 0^0 + \cdots + a_n 0^n = a_0 0^0,$$

and we are implicitly assuming that $0^0 = 1$.

Is $0^0 = 1$? 3

- We can also consider a power series like

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Then

$$f(0) = 1 = \sum_{n=0}^{\infty} 0^n = 0^0.$$

If we do not define 0^0 to be 1, we will have trouble with even simple expressions like this.

Is $0^0 = 1$? 4

- Another example is the Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

- Setting $a = 0$ on both sides and assuming $b \neq 0$ we get

$$b^n = (0+b)^n = \sum_{k=0}^{n} \binom{n}{k} 0^k b^{n-k} = \binom{n}{0} 0^0 b^n = 0^0 b^n,$$

where, we have used that $0^k = 0$ for $k > 0$, and that $\binom{n}{0} = 1$.

- We see that we must set $0^0 = 1$ in order for the binomial theorem to be valid.

Is $0^0 = 1$? 5

- Another reason why $0^0 = 1$ is because x^0 is the "empty product", which should be the multiplicative identigy 1. For the same reason we also get $0! = 1$.
- In order for the differentiation rule

$$\frac{d}{dx} x^n = n x^{n-1}$$

to hold for $n = 1$ when $x = 0$, we get

$$1 = \frac{d}{dx} x = \frac{d}{dx} x^1 = 1 \cdot x^{1-1} = x^0,$$

which requires $0^0 = 1$.

- So to sum up, we must write $0^0 = 1$ to make many expressions work.

Rational numbers

- We will study the rational numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

We want to show that $\sqrt{2}$ is irrational.

- We will need the following lemma

Lemma

A natural number a is even if and only if a^2 is even.

- Proof: If a is even we can write $a = 2k$ with $k \in \mathbb{Z}$ and then

$$a^2 = (2k)^2 = 4k^2 = 2(2k^2),$$

so we see that

$$a \text{ is even} \implies a^2 \text{ is even.}$$

Rational numbers 2

- In order to show

$$a \text{ is even} \impliedby a^2 \text{ is even,}$$

we will use that

$$p \implies q \quad \text{is the same as} \quad \neg p \impliedby \neg q.$$

- So we will show that

$$a \text{ is odd} \implies a^2 \text{ is odd.}$$

- If $a = 2k + 1$ with $k \in \mathbb{Z}$, then

$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

so a^2 is odd. $\square$

Rational numbers 3

- **Theorem**

  $\sqrt{2}$ is irrational.

  - Proof: We will assume that $\sqrt{2}$ is rational and can be written as a/b, where $a, b \in \mathbb{Z}$ are relatively prime, i.e., they have no common factors. Then

    $$2 = \frac{a^2}{b^2} \quad \text{and} \quad 2b^2 = a^2,$$

    and we see that a^2 is even. But then we know from the above Lemma that a is also even, so $a = 2k$ with $k \in \mathbb{Z}$ and

    $$a^2 = (2k)^2 = 4k^2 = 2(b^2) \quad \text{or} \quad b^2 = 2k^2.$$

    Since b^2 is even, it follows that b is also even. We have now shown that both a and b are even, but this contradicts the assumption that a and b are relatively prime. $\square$

Countable

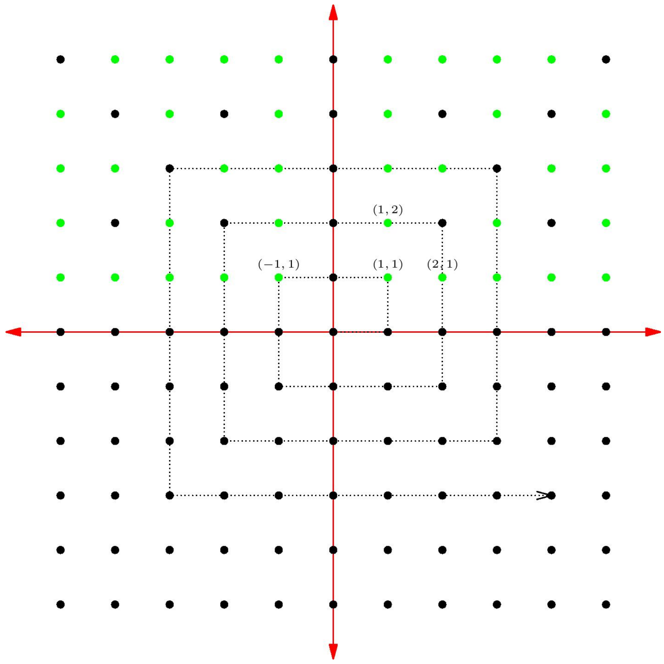
- We say that two sets X and Y have the same cardinality if there is a bijection $f\colon X \to Y$.
- We say that X is countably infinite if there is a bijection $f\colon \mathbb{N} \to X$.
- We say that X is countable if there is a surjection $f\colon \mathbb{N} \to X$. This means that we can write the elements of X as a list.
- A countable set is either countably infinite or finite.

Countable 2

- The set of integers, $\mathbb{Z}$ is countable, since
$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}.$$
- The set of rational numbers, $\mathbb{Q}$, is countable. This can be seen in many ways.

Countable 3

- The rational numbers a/b correspond to the pair (a, b), so $\mathbb{Q}$ corresponds to $\mathbb{Z} \times (\mathbb{Z} - \{0\})$.



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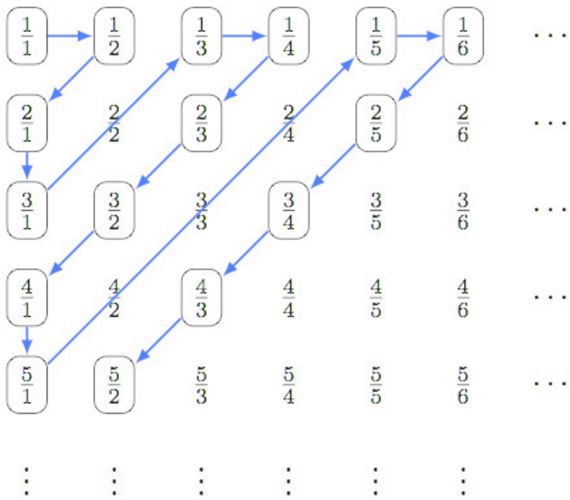
Countable 4



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- This shows that the set of positive rationals is countable. By alternating between positive and negative numbers, we can show that the whole of $\mathbb{Q}$ is countable.
- This picture also shows that a countable union of countable sets is countable.

Countable 5

- In 1874, Georg Cantor (1845 –- 1918) proved that $\mathbb{R}$ is not countable.
- Assume that $\mathbb{R}$ is countable. Then $[0, 1]$ is also countable, and we can write $[0, 1] = \{r_1, r_2, \ldots\}$ where $r_i = 0.d_{i1}d_{i2}\ldots$.

$$\begin{array}{l} r_1 = 0.\ d_{11}\ d_{12}\ d_{13}\ d_{14}\ d_{15}\ \cdots \\ r_2 = 0.\ d_{21}\ d_{22}\ d_{23}\ d_{24}\ d_{25}\ \cdots \\ r_3 = 0.\ d_{31}\ d_{32}\ d_{33}\ d_{34}\ d_{35}\ \cdots \\ r_4 = 0.\ d_{41}\ d_{42}\ d_{43}\ d_{44}\ d_{45}\ \cdots \\ r_5 = 0.\ d_{51}\ d_{52}\ d_{53}\ d_{54}\ d_{55}\ \cdots \\ \quad \vdots \end{array}$$

- $r = 0.\ d_1\ d_2\ d_3\ d_4\ d_5\ \cdots$
- We then construct a number $r = 0.d_1d_2\ldots$, where $d_i \neq d_{ii}$ and $d_i \neq 9$. Then $r \neq r_i$ for all i, and r is not in the list.