UiO : University of Oslo

Calculus and Counterexamples

Helmer Aslaksen

Dept. of Teacher Education & Dept. of Mathematics
University of Oslo

helmer.aslaksen@gmail.com
www.math.nus.edu.sg/aslaksen/

Limits in high school mathematics

- To differentiate polynomials, you only need algebra to compute limits.
- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.
- Definition of e.

Definition of e

- Does $s_n = \left(1 + \frac{1}{n}\right)^n$ converge?
- We want to use the fact that a bounded and increasing sequence converges, but it is not clear that s_n is either bounded or increasing.
- The binomial formula shows that

$$\begin{aligned}
s_n &= \left(1 + \frac{1}{n}\right)^n \\
&= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} \\
&\quad + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}\frac{1}{n^n} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\
&\quad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right).
\end{aligned}$$

Definition of e 2

- The product is hard to analyze, since the number of factors increase, while the factors themselves decrease. However, the binomial formula converts s_n to a sum of n terms.
- Since all the terms in the parenthesis are positive, we have now written s_n as a sum of n positive terms. When we go from s_n to s_{n+1}, the first n terms do not change, and we simply add another positive term. It is therefore clear that s_n is increasing.

Definition of e 3

- Consider the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ with partial sums
$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$
- Since t_n is obtained from s_n by removing the parenthesis, and all the terms in the parenthesis are less than 1, we see that $s_n \leq t_n$. Since going from t_n to t_{n+1} just adds a positive term, we see that t_n is also increasing.
- Since
$$n! = 1 \cdot 2 \cdot 3 \ldots n > 1 \cdot 2 \cdot 2 \ldots 2 = 2^{n-1},$$
we have
$$s_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.$$
- It follows that s_n is bounded and increasing, so e exists and $e \leq 3$.

Continuity

- $f\colon U \to \mathbb{R}$ is continuous at $a \in U$ if $\lim_{x\to a} f(x) = f(a)$ and continuous on U if it is continuous at all points in U.
- Some people say that f is continuous if and only if we can draw the graph of f without lifting the pen. However, $f(x) = 1/x$ is continuous on $U = \mathbb{R} - \{0\}$.

Product rule

- $$\begin{aligned} f(x+\Delta x)g(x+\Delta x) - f(x)g(x) &= (f(x+\Delta x) - f(x))g(x) \\ &+ (g(x+\Delta x) - g(x))f(x) \\ &+ (f(x+\Delta x) - f(x))(g(x+\Delta x) - g(x)) \end{aligned}$$

Source of counterexamples

- $$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
- f_0 is not continuous, f_1 $\lim_{x \to 0} f_1(x) = 1$

Source of counterexamples 2

- $$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
- $$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Monotonicity

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that
  $$\frac{f(b) - f(a)}{b - a} = f'(c).$$
- $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b).
- $f' \geq 0$ on $(a, b) \implies f$ is increasing on (a, b).
- $f' \geq 0$ on $(a, b) \impliedby f$ is increasing on (a, b).
- $f(x) = x^3$ shows that $f' \geq 0$ on $(a, b) \not\Longleftarrow f$ is strictly increasing on (a, b).

Extreme point 1

- If c is an extreme point and $f'(c)$ exists, then $f'(c) = 0$.
- First Derivative Test: If f' exists around c, and f' changes sign at c, then c is an extreme point.
- Second Derivative Test: If $f'(c) = 0$ and $f''(c)$ is positive (negative), then c is a minimum (maximum).

Extreme point 2

- If f' changes sign at c, then c is an extreme point. The converse is not always true.
- $f(x) = x^2(2 + \sin(1/x))$, $f'(x) = 4x + 2x\sin(1/x) - \cos(1/x)$.
- $x^2 + x^2\sin(1/x))$ has infinitely many zeros.
- If f' is positive on (a, b), then f is increasing on (a, b). But what if we only know that $f'(c) > 0$? Can we say that f is increasing on an interval around c?
- $f(x) = x + 2x^2\sin(1/x)$, $f'(x) = 1 + 4x\sin(1/x) - 2\cos(1/x)$ is both positive and negative in every neighborhood of 0.

Point of inflection

- We say that c is a point of inflection if f has a tangent line at c and f'' changes sign at c. (Some people only require that f should be continuous at c.)
- $f(x) = x^3$ has $f'(0) = 0$, but 0 is not an extremum, but a point of inflection.
- $f(x) = x^3 + x$ shows that f' does not have to be 0 at a point of inflection.

Point of inflection 2

- $f(x) = x^{1/3}$ has a point of inflection at 0, has a tangent line at 0, but $f'(0)$ and $f''(0)$ do not exist. (Vertical tangent line. Just bend a bit, and you get a point of inflection.)
- $$f(x) = \begin{cases} x^2 \text{ if } x \geq 0, \\ -x^2 \text{ if } x < 0, \end{cases}$$

  has a point of inflection at 0, and $f'(0)$ exists, but $f''(0)$ does not exist. (First derivatives match, so we get a tangent line, but second derivatives do not match.)

Point of inflection 3

1. If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.
2. If c is a point of inflection, then c is an isolated extremum of f'.
3. If c is a point of inflection, then the curve lies on different sides of the tangent line at c.

Point of inflection 4

- Proof of 3: We use MVT go get x_1 between c and x with

$$\frac{f(x) - f(c)}{x - c} = f'(x_1),$$

or

$$f(x) = f(c) + f'(x_1)(x - c).$$

- We now use MVT again to get x_2 between c and x_1 with

$$\frac{f'(x_1) - f'(c)}{x_1 - c} = f''(x_2),$$

or

$$f'(x_1) = f'(c) + f''(x_2)(x_1 - c).$$

- Combining this, we get

$$\begin{aligned} f(x) &= f(c) + f'(x_1)(x - c) \\ &= f(c) + f'(c)(x - c) + f''(x_2)(x - c)(x_1 - c). \end{aligned}$$

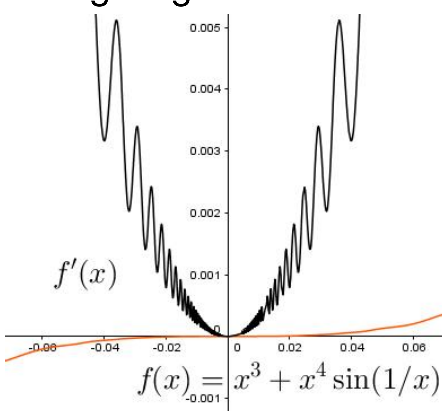
Point of inflection 5

- The tangent line to $f(x)$ at c is $t(x) = f(c) + f'(c)(x - c)$, so the distance between f and the tangent is $f'(x_2)(x - c)(x_1 - c)$.
- Since $(x_1 - c)$ and $(x_1 - c)$ have the same sign, their product is positive. But $f''(x)$ changes sign at c, so $f(x)$ will lie on different sides of the tangent at c.

Point of inflection 6

- Converse to 1 is false: $f(x) = x^4$ has $f''(0) = 0$, but $f''(x) \geq 0$.
- Converse to 2 is false: $f(x) = x^3 + x^4 \sin(1/x)$ has

$$\begin{aligned} f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x) \geq 0 \end{aligned}$$

  in a neighborhood of 0, so 0 is an isolated minimum of $f'(x)$. We have $f''(0) = 0$, but $f''(x) = 6x - \sin(1/x) - 6x \cos(1/x) + 12x^2 \sin(1/x)$ does not change sign.
- 

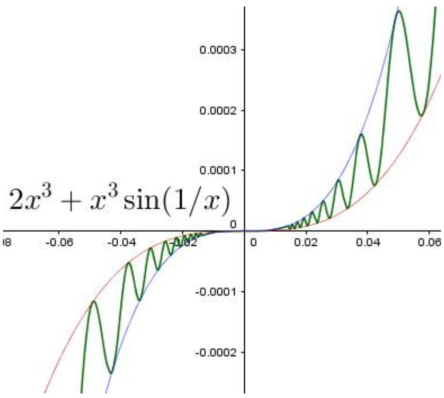
Point of inflection 7

- We need to “integrate” the example $2x^2 + x^2 \sin(1/x)$. Since the derivative of $1/x$ is $-1/x^2$, we try

$$\begin{aligned} f(x) &= x^3 + x^4 \sin(1/x), \\ f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x)). \end{aligned}$$

- The first two terms give us the shape we want, and the last terms is so small that we can ignore it.

Point of inflection 8

- Converse to 3 is false: $f(x) = 2x^3 + x^3 \sin(1/x) = x^3(2 + \sin(1/x))$ lies below the tangent ($y = 0$) on one side and above the tangent on another, but $f''(x) = 12x + 6x \sin(1/x) - 4\cos(1/x) - (1/x)\sin(1/x)$ does not change sign, since when x is small, the last term will be oscillate wildly.
- The cubic terms gives the desired shape of the curve, and since the derivative of $1/x$ is $-1/x^2$, we will get a term of the form $(1/x)\sin(1/x)$ in $f''(x)$, which will make it oscillate wildly.
- 

L'Hôpital's Rule

- Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,$$

  for $L \in \mathbb{R} \cup \infty$.

- Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,$$

  for $L \in \mathbb{R} \cup \infty$.

L'Hôpital's Rule 2

- L'Hôpital does not say that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \impliedby \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

- If $f(x) = x + \sin x$ and $g(x) = x$, then

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1 + \cos x}{1}$$

  does not exist, while

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \left(1 + \frac{\sin x}{x}\right) = 1.$$