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Mean and Extreme

Helmer Aslaksen

Dept. of Teacher Education & Dept. of Mathematics
University of Oslo

helmer.aslaksen@gmail.com
www.math.nus.edu.sg/aslaksen/

What kind of topics will I cover?

- Some topics you can take straight to the classroom.
- Some topics will help you answer questions that you may be asked once in a while by strong pupils.
- Some topics you will probably never discuss with any pupils, but knowing it will help your own understanding of the topics.
- There should be a line of sight back towards school mathematics.

Arithmetic series

- Why is an arithmetic series called arithmetic?
- We define the arithmetic mean to be

$$\mathrm{AM}(x, y) = \frac{1}{2}(x + y).$$

- In an arithmetic series, every term is the arithmetic mean of the two surrounding terms.

$$\frac{1}{2}(a_{n+1} + a_{n-1}) = \frac{1}{2}(a_n + d + a_n - d) = \frac{1}{2}2a_n = a_n.$$

Geometric series

- Why is a geometric series called geometric?
- We define the geometric mean to be

$$\mathrm{GM}(x, y) = (xy)^{1/2}.$$

- In a geometric series (with positive terms), every term is the geometric mean of the two surrounding terms.
- $$(a_{n+1}a_{n-1})^{1/2} = [(a_n r)(a_n/r)]^{1/2} = (a_n^2)^{1/2} = a_n.$$

Harmonic series

- Why is the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

called harmonic?

- We define the harmonic mean to be

$$\mathrm{HM}(x, y) = \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$

- In the harmonic series, every term is the harmonic mean of the two surrounding terms.
- $$\frac{2}{\frac{1}{1/(n+1)} + \frac{1}{1/(n-1)}} = \frac{2}{n+1+n-1} = \frac{2}{2n} = \frac{1}{n}.$$

Confusing means — Simpson's paradox

- In 1973 UC Berkeley admitted 44% of males and 35% of females who applied to grad school. The tables show admission data from the six largest departments.

Department	Male acceptance rate	Female acceptance rate
A	62%	82%
B	63%	68%
C	37%	34%
D	33%	35%
E	28%	24%
F	6%	7%

Department	Male		Female	
	Applicants	%	Applicants	%
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	373	6%	341	7%

Confusing means — Simpson's paradox 2

- You teach a class with 20 strong students and 5 weak students. At the final exam, the strong students get an average of 80 points and the weak students get an average of 50.
- Your Principal is impressed that all the weak students passed, and next year you get a class with 15 strong students and 10 weak students. This year the strong students increase their average to 85, and the weak students increase their average to 55.
- You are quite proud of yourself, but the Principal calls you in and is unhappy because the overall average has dropped from $(20 \cdot 80 + 5 \cdot 50)/25 = 74$ to $(15 \cdot 85 + 10 \cdot 55)/25 = 73$.
- The arithmetic mean may look innocent, but can be devious.

What is the meaning of the geometric mean?

- Given a rectangle with sides x and y, we want to find a square with the same area. What is the side of the square?
- $z = \sqrt{xy} = \mathrm{GM}(x, y)$.

What is the meaning of the harmonic mean?

- You drive to work during rush hour with an average speed of 30km/h. Going home you manage an average speed of 60km/h. What was your average speed for the whole trip?
- Assume that the distance is d. Then your average speed was

$$\frac{2d}{d/30 + d/60} = \frac{2}{1/30 + 1/60} = \frac{2 \cdot 60}{2 + 1} = 40 = H(30, 60).$$

- The term harmonic is related to music theory.

The harmonic series diverges

- This was shown by Nicole Oresme around 1350.

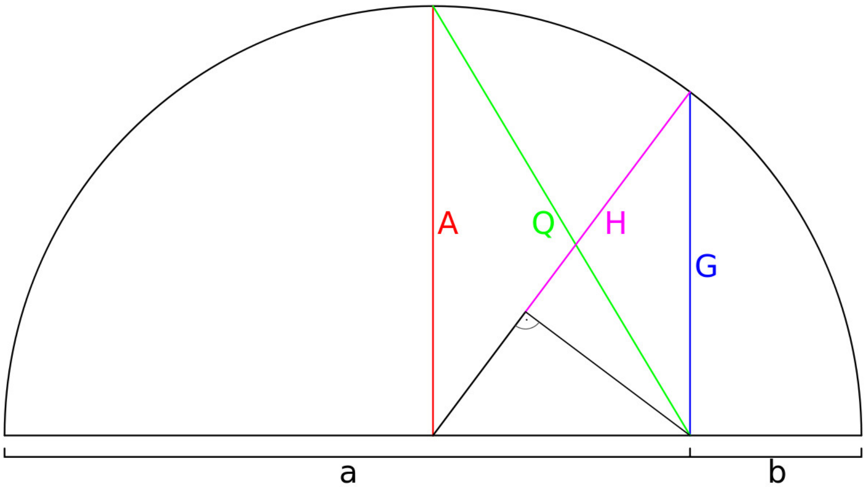
$$\begin{aligned} &1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots \\ > &1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots \end{aligned}$$

- This argument shows that

$$\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{k}{2},$$

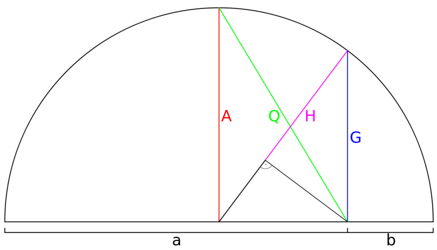
and we see that the series diverges.

The AGH inequality



$$\mathrm{AM}(a,b) > \mathrm{GM}(a,b) > \mathrm{HM}(a,b).$$

Proof of the AGH inequality



$$\left(\frac{a+b}{2}\right)^2 = G^2 + \left(a - \frac{a+b}{2}\right)^2$$

$$\frac{(a+b)^2}{4} = G^2 + \frac{(a-b)^2}{4}$$

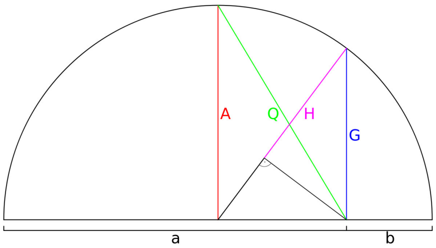
$$(a+b)^2 = 4G^2 + (a-b)^2$$

$$2ab = 4G^2 - 2ab$$

$$G^2 = ab$$

$$G = \sqrt{ab} = \mathrm{GM}(a,b).$$

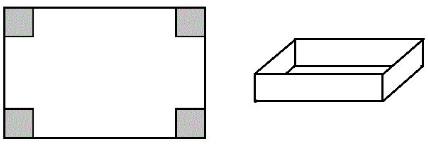
Proof of the AGH inequality 2

- 

- By similar triangles $A/G = G/H$ or $G^2 = AH$. Hence

$$H = \frac{G^2}{A} = \frac{ab}{\frac{a+b}{2}} = \frac{2}{\frac{a+b}{ab}} = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \mathrm{HM}(a,b).$$

Maximum volume of a cut off box

- 
- Consider a rectangle of height 1 and width L. We cut off squares of side x in each corner and fold to get a box of volume

$$V(L,x) = x(L-2x)(1-2x).$$

- If we solve $V'(x) = 0$ we get

$$x = \frac{L+1-\sqrt{(L+1)^2-3L}}{6}.$$

- If we set $L = 1$, we get $x = \frac{1+1-\sqrt{(1+1)^2-3\cdot 1}}{6} = \frac{2-\sqrt{2^2-3}}{6} = \frac{1}{6}$.

Maximum volume of a cut off box 2

- In that case the area of the base is $(2/3)^2 = 4/9$, while the area of the side wall equals $4 \cdot 1/6 \cdot 2/3 = 4/9$.
- Is it a coincidence that these two areas are equal?
- Consider a convex, closed curve W, and let $W(t)$ be the curve obtained by pushing W inward along the normal line a distance t. We can then "fold" up to get a box.
- Let $A(t)$ be the area of the region inside $W(t)$, let $P(t)$ be the perimeter of $W(t)$ and let $V(t)$ be the volume of the box.
- I claim that $A'(t) = -P(t)$.

Maximum volume of a cut off box 3

- We have $A'(t) = \lim_{h\to 0} \frac{A(t+h)-A(t)}{h}$, and we can interpret $A(t+h) - A(t)$ as the negative of the area of a "ring" of thickness h. Since the area of the ring will have area approximately equal to $P(t)t$, we get that

$$A'(t) = \lim_{h\to 0} \frac{A(t+h) - A(t)}{h} \approx \lim_{h\to 0} \frac{-P(t)h}{h} = -P(t).$$

- We have $V(t) = A(t)t$, so $V'(t) = A'(t)t + A(t) = -P(t)t + A(t) = 0$ precisely when the area of the base equals the area of the wall.

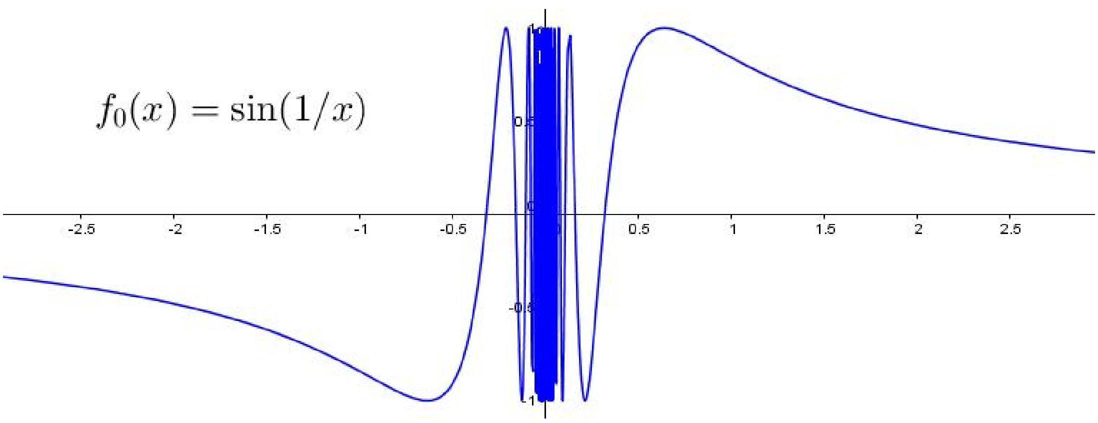
Source of counterexamples

- $$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
- Note that

$$\lim_{x \to \infty} x \sin(1/x) = \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{y \to 0} \frac{\sin(y)}{y} = 1.$$

$\sin(1/x)$

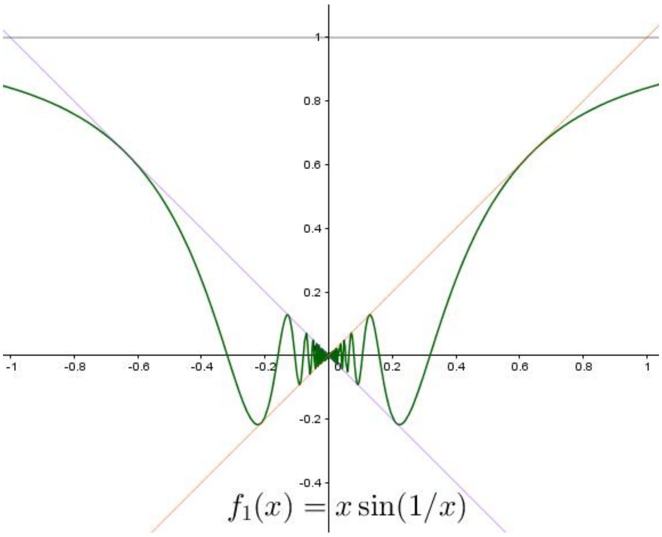
- $$f_0(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



- f_0 is not continuous at $x = 0$, since $\lim_{x \to 0} f_0(x)$ does not exist.

$x \sin(1/x)$

- $$f_1(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



- Remember that $\lim_{x\to\infty} f_1(x) = 1$.

$x \sin(1/x)$ part 2

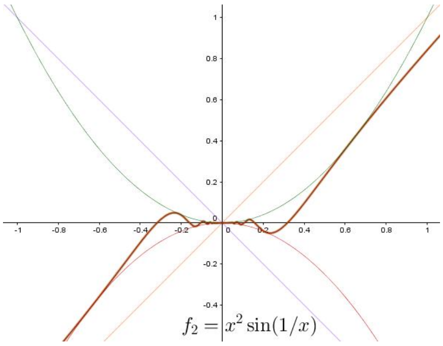
- f_1 is continuous, since it is squeezed by $\pm x$, but

$$\lim_{x\to 0} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} \sin(1/x),$$

does not exist, so f_1 is not differentiable at $x = 0$.

$x^2 \sin(1/x)$

- $$f_2(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



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- Setting $y = 1/x$ and using L'Hôpital's rule, we get
$$\lim_{x \to \infty} (x^2 \sin(1/x) - x) = \lim_{y \to 0} (\frac{\sin y}{y^2} - \frac{1}{y}) =$$
$$\lim_{y \to 0} \frac{\sin y - y}{y^2} = \lim_{y \to 0} \frac{\cos y - 1}{2y} = \lim_{y \to 0} \frac{-\sin y}{2} = 0.$$
- f_2 is differentiable, since it is squeezed by $\pm x^2$.

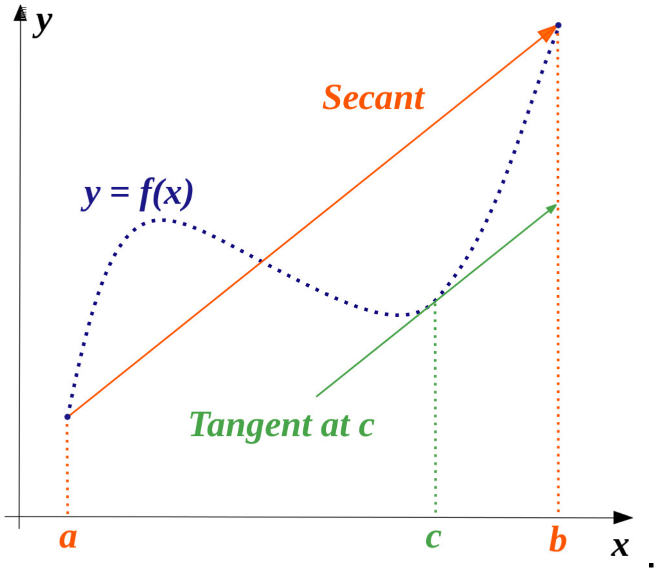
$x^2 \sin(1/x)$ part 2

- $$f_2'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin(1/x) = 0.$$
However, for $x \neq 0$ we have $f_2'(x) = 2x \sin(1/x) - \cos(1/x)$, and
$$\lim_{x \to 0} f_2'(x) = \lim_{x \to 0} (2x \sin(1/x) - \cos(1/x))$$
does not exist.
- So f_2 is differentiable, but not continuously differentiable!
- This is the mother of all counterexamples!

Monotonicity

- Mean Value Theorem: Assume that f is differentiable on (a, b) and continuous on $[a, b]$. Then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Monotonicity 2

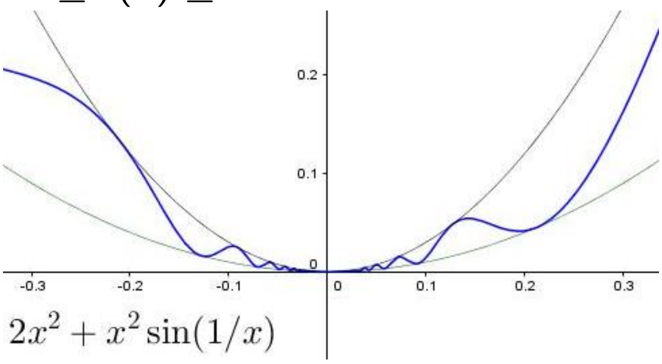
- f is increasing if $x < y \implies f(x) \leq f(y)$.
- f is strictly increasing if $x < y \implies f(x) < f(y)$.
- Assume that $f' > 0$ on (a, b). Given $a < x < y < b$, we can find $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) > 0$. It follows that
- $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b).
- $f' \geq 0$ on $(a, b) \implies f$ is increasing on (a, b).
- If f is increasing, then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0$. It follows that
- $f' \geq 0$ on $(a, b) \impliedby f$ is increasing on (a, b).
- $f(x) = x^3$ shows that $f' > 0$ on $(a, b) \not\Longleftarrow f$ is strictly increasing on (a, b).
- Limits do not preserve strict inequalities.

Extreme point

- Assume that c is a minimum point and that $f'(c)$ exists. Consider $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$. If h is positive, the fraction is positive, and if h is negative, the fraction is negative. Since the limit exists, it must be zero.
- Assume that f' exists around c, and $f'(x)$ is positive for $x > c$ and negative for $x < c$. If $x > c$, then there is a d between c and x such that $f(x) - f(c) = f'(d)(x - c) > 0$. If $x < c$, then there is a d between x and c such that $f(c) - f(x) = f'(d)(c - x) < 0$. It follows that c is a minimum point.
- However, the converse is not always true.

Extreme point 2

- We start with a parabola and add $x^2 \sin(1/x)$ to create an oscillating parabola.
- Since $x^2 + x^2 \sin(1/x))$ has infinitely many zeros, we instead start with $2x^2$ and use $f(x) = x^2(2 + \sin(1/x))$, which satisfies $x^2 \geq f(x) \geq 3x^2$.



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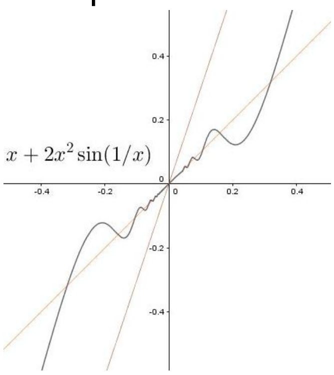
Extreme point 3

- f obviously has a minimum at $x = 0$, but it is easy to see that f' is both positive and negative arbitrarily close to $x = 0$.
- We have $f'(x) = 4x + 2x\sin(1/x) - \cos(1/x)$, and if x is close to zero, the first two terms will be close to zero, too, while the last term will oscillate between 1 and -1.

Increasing

- If f' is positive on (a, b), then f is increasing on (a, b). But what if we only know that $f'(c) > 0$? Can we say that f is increasing on an interval around c?
- We start with a straight line and add $x^2\sin(1/x)$ to create an oscillating line. It turns out that it will be easer if we add $2x^2\sin(1/x)$, so we set $f(x) = x + 2x^2\sin(1/x)$.
- Then $f'(x) = 1 + 4x\sin(1/x) - 2\cos(1/x)$, and when x is close to zero, then will oscillate between 3 and -1, so f' will be both positive and negative in every neighborhood of 0.



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Point of inflection

- We say that c is a point of inflection if f has a tangent line at c and f'' changes sign at c. (Some people only require that f should be continuous at c.)
- Let us consider some examples.
- $f(x) = x^3$ has $f'(0) = 0$, but 0 is not an extremum, but a point of inflection.
- $f(x) = x^3 + x$ shows that f' does not have to be 0 at a point of inflection.

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Point of inflection 2

- $f(x) = x^{1/3}$ has a point of inflection at 0, has a tangent line at 0, but $f'(0)$ and $f''(0)$ do not exist. (Vertical tangent line. Just bend a bit, and both derivatives will exist.)
- $$f(x) = \begin{cases} x^2 \text{ if } x \geq 0, \\ -x^2 \text{ if } x < 0 \end{cases}$$
  has a point of inflection at 0, and $f'(0)$ exists, but $f''(0)$ does not exist. (First derivatives match, so we get a tangent line, but second derivatives do not match.)
- $$f(x) = \begin{cases} x^2 + x \text{ if } x \geq 0, \\ -x^2 - 2x \text{ if } x < 0 \end{cases}$$
  does not have a tangent line at 0, since the first derivatives do not match. However, the second derivative changes sign at 0. Is this a point of inflection? I have chosen to not include this, but some people do.

Point of inflection 3

1. If c is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$.
2. If c is a point of inflection, then c is an isolated extremum of f'.
3. If c is a point of inflection, then the curve lies on different sides of the tangent line at c.

Point of inflection 4

- Proof of 3: We use MVT go get x_1 between c and x with
$$\frac{f(x) - f(c)}{x - c} = f'(x_1),$$
or
$$f(x) = f(c) + f'(x_1)(x - c).$$
- We now use MVT again to get x_2 between c and x_1 with
$$\frac{f'(x_1) - f'(c)}{x_1 - c} = f''(x_2),$$
or
$$f'(x_1) = f'(c) + f''(x_2)(x_1 - c).$$
- Combining this, we get
$$\begin{aligned} f(x) &= f(c) + f'(x_1)(x - c) \\ &= f(c) + f'(c)(x - c) + f''(x_2)(x - c)(x_1 - c). \end{aligned}$$

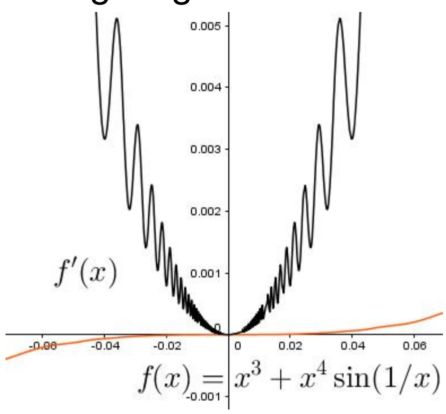
Point of inflection 5

- The tangent line to $f(x)$ at c is $t(x) = f(c) + f'(c)(x - c)$, so the distance between f and the tangent is $f'(x_2)(x - c)(x_1 - c)$.
- Since $(x_1 - c)$ and $(x_1 - c)$ have the same sign, their product is positive. But $f''(x)$ changes sign at c, so $f(x)$ will lie on different sides of the tangent at c.

Point of inflection 6

- Converse to 1 is false: $f(x) = x^4$ has $f''(0) = 0$, but $f''(x) \geq 0$.
- Converse to 2 is false: $f(x) = x^3 + x^4 \sin(1/x)$ has

$$\begin{aligned} f'(x) &= 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x) \\ &= x^2(3 - \cos(1/x) + 4x \sin(1/x) \geq 0 \end{aligned}$$

  in a neighborhood of 0, so 0 is an isolated minimum of $f'(x)$. We have $f''(0) = 0$, but $f''(x) = 6x - \sin(1/x) - 6x \cos(1/x) + 12x^2 \sin(1/x)$ does not change sign.
- 

Point of inflection 7

- We need to “integrate” the example $2x^2 + x^2 \sin(1/x)$. Since the derivative of $1/x$ is $-1/x^2$, we try

$$f(x) = x^3 + x^4 \sin(1/x),$$
$$f'(x) = 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x)$$
$$= x^2(3 - \cos(1/x) + 4x \sin(1/x)).$$

- The first two terms give us the shape we want, and the last terms is so small that we can ignore it.

Point of inflection 8

- Converse to 3 is false:
  $f(x) = 2x^3 + x^3 \sin(1/x) = x^3(2 + \sin(1/x))$ lies below the tangent ($y = 0$) on one side and above the tangent on another, but $f''(x) = 12x + 6x \sin(1/x) - 4\cos(1/x) - (1/x)\sin(1/x)$ does not change sign, since when x is small, the last term will be oscillate wildly.
- The cubic terms gives the desired shape of the curve, and since the derivative of $1/x$ is $-1/x^2$, we will get a term of the form $(1/x)\sin(1/x)$ in $f''(x)$, which will make it oscillate wildly.
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