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Solid Geometry

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Parametric equations 1

- We use two types of equations in solid geometry: Parametric equations and Cartesian (or coordinate) equations.
- A parametric equation of a line in $\mathbb{R}^n$ is of the form $\vec{X} = \vec{P} + t\vec{V}$, where $\vec{X} = \{x_1, \ldots, x_n\}$ is an arbitrary point on the line, $\vec{P}$ is a given point on the line, and $\vec{V}$ is a direction vector of the line.
- A parametric equation of a plane in $\mathbb{R}^n$ is of the form $\vec{X} = \vec{P} + s\vec{U} + t\vec{V}$, where $\vec{X} = \{x_1, \ldots, x_n\}$ is an arbitrary point on the plane, $\vec{P}$ is a given point on the line, and $\vec{U}$ and $\vec{V}$ are spanning vectors of the plane.

Parametric equations 2

- Notice that we only need one equation to describe either of these objects. However, if we write the equations componentwise, we will get n equations. For example, we can write the equation for a line in $\mathbb{R}^3$ as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

  or as

$$\begin{aligned} x &= p_1 + tv_1 \\ y &= p_2 + tv_2 \\ z &= p_3 + tv_3. \end{aligned}$$

Cartesian equations 1

- When we use parametric equations, the dimension of the ambient space is clear from the size of the vectors. The equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

  is a line in $\mathbb{R}^2$, while the equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

  is a line in $\mathbb{R}^3$.
- However, the Cartesian equation $x = 0$ is a point if we are in $\mathbb{R}$, the y-axis if we are in $\mathbb{R}^2$ and the y-z plane if we are in $\mathbb{R}^3$. So you need to know both the equation and the ambient space.

Cartesian equations 2

- The Cartesian equation for a line in $\mathbb{R}^2$ is of the form $ax + by + c = 0$.
- We start with two degrees of freedom in $\mathbb{R}^2$, so with one restriction, we get an object of dimension 1 and codimension 1. The codimension of an object is the dimension of the ambient space minus the dimension of the object.
- However, the corresponding equation in $\mathbb{R}^3$, $ax + by + cz + d = 0$, gives us an object in $\mathbb{R}^3$ of codimension 1 (since we have one equation), namely a plane.
- So how do we get the Cartesian equation of a line in $\mathbb{R}^3$?
- A line in $\mathbb{R}^3$ can be written as the intersection of two planes (in infinitely many ways). So the Cartesian equation of a line in $\mathbb{R}^3$ will be of the form

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0. \end{aligned}$$

Comparing Parametric and Cartesian equations

- Notice that parametric equations build objects by adding more spanning vectors, but still just using one equation. If we wanted to describe a 4-dimensional plane in $\mathbb{R}^6$, the parametric equation would be of the form

$$\vec{X} = \vec{P} + t_1\vec{V}_1 + \cdots + t_4\vec{V}_4.$$

- However, the Cartesian description would be a system of two equations, since the codimension is 2.