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Trigonometry

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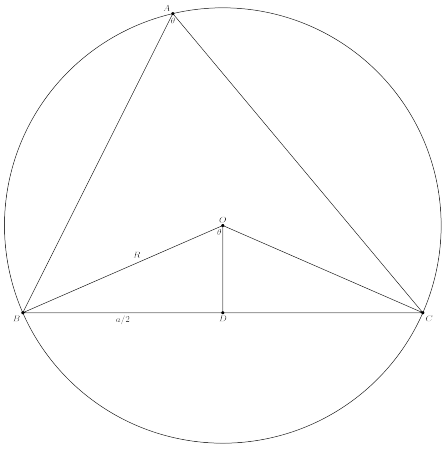
Extended Law of Sines



- Let R be the radius of the circumscribed circle. Consider the triangle ΔBDO. Then

$$\sin\theta = \frac{a/2}{R}, \quad \text{so} \quad \frac{a}{\sin\theta} = 2R.$$

The Law of Cosines

- $$a^2 + b^2 - 2ab\cos C = c^2.$$

Laws of Trigonometry

- Since sine is not injective on $[0, \pi]$, the law of sines may give two possible values if you are trying to determine an angle that could be the largest angle. If you know that the angle is not the largest, you are OK, since there can be at most one obtuse angle.
- The law of cosines may give two possible values if you are trying to determine a side that is adjacent to the angle, since the quadratic equation then has a first order term and may have two positive solutions. If you use it to determine an angle you are OK since cosine is injective on $[0, \pi]$. If you know two sides and the angle between them, you are OK, since you get a quadratic equation without first order term, which therefore only has one positive solution.
- If you have a choice, the law of sines is easier to use computationally.

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Laws of Trigonometry 2

- The law of sines requires that you know one “pair” of an angle and its opposite side, and one single side or angle. Think of the law of sines as a single friend going out with a couple.
- The law of cosines requires that you either know two sides and the angle between them, and then find the last side, or that you know three sides and then find an angle, or that you know two sides and the angle opposite one of them. In the first two cases, you can think of the law of cosines as three single friends going out, while the last case is similar the law of sines.
- This raises several questions. If you know a pair and a single side, should you use the law of sines or the law of cosines? If you know three sides, does it matter which angle you apply the law of cosines to? If you know three sides and one angle, does it matter which angle you apply the law of sines to?

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Solving a Triangle

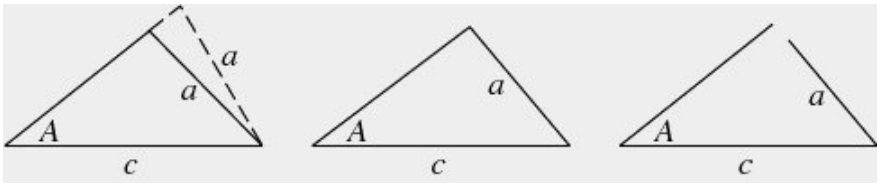
SSS Use the law of cosines for the *largest* angle (the anlge opposite the *longest* side), use the law of sines for another angle, and then use the sum of the angles theorem. (Remember that cosine is a *long* word.)

SAS Use the law of cosines for the opposite side, the law of sines for the *smaller* of the two remaining angles, and the sum of the angles theorem.

SsA Use the law of sines for the angle opposite the *short* side, use the sum of the angles theorem, and then the law of sines for the last side. (Remember that sine is a *short* word.)

ASA Use the sum of the angles theorem, and then use the law of sines twice. Remember that [ASA] really is [AAAS], so it doesn’t matter how the side is placed with respect to the two given angles.

sSA

- 

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$$
\begin{aligned}
a^2 &= x^2 + c^2 - 2xc\cos A \\
x^2 &- (2c\cos A)x + c^2 - a^2 = 0 \\
x &= \frac{2c\cos A \pm \sqrt{4c^2\cos^2 A - 4(c^2 - a^2)}}{2} \\
x &= c\cos A \pm \sqrt{c^2(\cos^2 A - 1) + a^2} \\
x &= c\cos A \pm \sqrt{a^2 - c^2\sin^2 A}
\end{aligned}
$$

sSA 2

- If A is acute, then the distance from B to the opposite side is $c\sin A$. If $a < c\sin A$, there will be no solutions, if $a = c\sin A$, then there will be one solution, if $c\sin A < a < c$, there will be two solutions, and if $c \leq a$ there will be one solution, since one intersection is to the left of A.
- This also follows from the law of cosines.

$$
\begin{aligned}
c\cos A - \sqrt{a^2 - c^2\sin^2 A} &= 0, \\
c^2\cos^2 A &= a^2 - c^2\sin^2 A, \\
c^2 &= a^2.
\end{aligned}
$$

sSA 3

- If A is obtuse, then there is no solution if $a \leq c$, and one solution if $c \leq a$.
- This also follows from the law of cosines, since in that case $c \cos A$ is negative, and we can only get a positive solution if the root is positive.

Laws of Trigonometry from an advanced viewpoint (optional)

- You can associate six variables to a triangle, namely the three sides and the three angles. However, the congruence class of a triangle is determined by only three variables, namely SSS, SsA, SAS, or ASA. That means that there should be at least three equations relating the six variables.
- We have six such equations, namely the three laws of cosines, the two laws of sines and the sum of the angles theorem. Are there any relationships between these laws?
- It turns out that you can deduce the laws of sines from the laws of cosines but you cannot deduce the the laws of cosines from the laws of sines, since there are only two of the latter.