Assignment 3 for MAT4170
Spline methods, Spring 2022

To be completed by Tuesday 29 March. Please send your solution as a single pdf file (including plots/figures) to `michaelf@math.uio.no`.

1. Suppose that x_0, x_1, x_2 are distinct, and let $f_i = f(x_i)$, $i = 0, 1, 2$, for some function f. Show by direct calculation that the recursive formula

$$[x_0, x_1, x_2]f = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0}$$

   can be expressed as

$$[x_0, x_1, x_2]f = \sum_{i=0}^{2} \frac{f_i}{\prod_{j \neq i}(x_i - x_j)}.$$

2. Show that if f is a polynomial of degree $\leq k - 1$ then $[x_0, \ldots, x_k]f = 0$.

3. Show that if $f(x) = 1/x$ and that $x_0, x_1, \ldots, x_k \neq 0$ then

$$[x_0, \ldots, x_k]f = \frac{(-1)^k}{x_0 x_1 \cdots x_k}.$$

   Hint: consider the divided difference of the product fg where $g(x) = x$.

4. Implement Boehm's algorithm and test it on an example. Verify graphically that the control polygons converge to the spline function as more and more knots are inserted. The plots should look like Fig. 4.5 in the compendium (include control polygons).

5. Optional. Prove the Leibniz rule for divided differences:

$$[x_0, x_1, \ldots, x_k](fg) = \sum_{i=0}^{k} ([x_0, \ldots, x_i]f)([x_i, \ldots, x_k]g).$$

Hint: let p and q be the polynomials of degree $\leq k$ that interpolate f and g respectively at $x_0, x_1, \ldots, x_k$, and express p and q as

$$p(x) = \sum_{i=0}^{k} (x - x_0) \cdots (x - x_{i-1})[x_0, \ldots, x_i]f,$$

$$q(x) = \sum_{j=0}^{k} (x - x_{j+1}) \cdots (x - x_k)[x_j, \ldots, x_k]g.$$

Now consider the polynomial pq.