Marsden's identity and linear independence of B-splines

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These notes derive Marsden's identity and use it to express polynomials in terms of B-splines and to show that B-splines are linearly independent.

1 Recursions

Let us recall the two kinds of recursion for spline functions. For any integers $d \geq 0$ and $n \geq 1$, let $\mathbf{t} = (t_1, t_2, \ldots, t_{n+d+1})$ be a non-decreasing knot vector. Such a sequence of knots together with a sequence of coefficients $c_j \in \mathbb{R}$, $j = 1, \ldots, n$, define a spline function

$$s(x) = \sum_{j=1}^{n} c_j B_{j,d}(x), \qquad x \in [t_{d+1}, t_{n+1}], \tag{1}$$

where the functions $B_{j,d}$ are B-splines. These B-splines satisfy a recursion. When $d = 0$,

$$B_{j,0}(x) = \begin{cases} 1 & x \in [t_j, t_{j+1}); \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and for $d \geq 1$,

$$B_{j,d}(x) = \frac{x - t_j}{t_{j+d} - t_j} B_{j,d-1}(x) + \frac{t_{j+d+1} - x}{t_{j+d+1} - t_{j+1}} B_{j+1,d-1}(x). \tag{3}$$

Let us consider how to compute the value $s(x)$, given some $x \in [t_{d+1}, t_{n+1}]$. First we locate the index μ such that $x \in [t_\mu, t_{\mu+1}]$. Then $s(x)$ is given by the local summation,

$$s(x) = \sum_{j=\mu-d}^{\mu} c_j B_{j,d}(x) \tag{4}$$

because all the B-splines other than $B_{\mu-d,d}, \dots, B_{\mu,d}$ are zero at x. Then there are two ways of evaluating s at x, i.e, calculating the value $s(x)$.

1.1 Algorithm 1

The first algorithm is to use the B-spline recursion directly to compute the $d+1$ values $B_{j,d}(x)$, $j = \mu - d, \dots, d$, and then to multiply them by the coefficients c_j, $j = \mu - d, \dots, \mu$ and sum them up. The recursion formula (3) gives us a triangular scheme for computing the B-splines. We fix $x \in [t_\mu, t_{\mu+1})$ and initialize the scheme by setting $B_{\mu,0} = 1$. Then, for $r = 1, 2, \dots, d$, and $j = \mu - r, \dots, \mu$, we set

$$B_{j,r} = \frac{x - t_j}{t_{j+d-r+1} - t_j} B_{j,r-1} + \frac{t_{j+d-r+1} - x}{t_{j+d-r+1} - t_j} B_{j+1,r-1}.$$

Here, we are using the fact that both $B_{\mu-r,r-1}$ and $B_{\mu+1,r-1}$ are zero at x. The flow of computations is as follows, where in each column, each value is computed from two values from the previous column.

$$\begin{array}{ccccc}
B_{\mu,0} & B_{\mu-1,1} & B_{\mu-2,2} & \cdots & B_{\mu-d,d} \\
 & B_{\mu,1} & B_{\mu-1,2} & \cdots & B_{\mu-d+1,d} \\
 & & B_{\mu,2} & \cdots & B_{\mu-d+2,d} \\
 & & & \ddots & \vdots \\
 & & & & B_{\mu,d}
\end{array}$$

1.2 Algorithm 2

Alternatively, we can use recursion on the coefficients c_j in (4). We fix x and initialize the algorithm by setting $c_j^0 = c_j$, $j = \mu - d, \dots, \mu$. Then for $r = 1, \dots, d$, and $j = \mu - d + r, \dots, \mu$, we set

$$c_j^r = \frac{t_{j+d-r+1} - x}{t_{j+d-r+1} - t_j} c_{j-1}^{r-1} + \frac{x - t_j}{t_{j+d-r+1} - t_j} c_j^{r-1}. \tag{5}$$

Theorem 1 *The last value computed, c_μ^d, is the value of s at x in (4).*

Proof. To prove this, consider the first step of the algorithm. By the B-spline recurrence for the $B_{j,d}$ we have

$$\begin{aligned} s(x) &= \sum_{j=\mu-d}^{\mu} c_j^0 \left(\frac{x - t_j}{t_{j+d} - t_j} B_{j,d-1}(x) + \frac{t_{j+d+1} - x}{t_{j+d+1} - t_{j+1}} B_{j+1,d-1}(x) \right) \\ &= \sum_{j=\mu-d+1}^{\mu} \left(\frac{t_{j+d} - x}{t_{j+d} - t_j} c_{j-1}^0 + \frac{x - t_j}{t_{j+d} - t_j} c_j^0 \right) B_{j,d-1}(x), \end{aligned}$$

where we have used the fact that both $B_{\mu-d,d-1}$ and $B_{\mu+1,d-1}$ are zero at x. Hence by the definition of c_j^1 in (5),

$$s(x) = \sum_{j=\mu-d+1}^{\mu} c_j^1 B_{j,d-1}(x).$$

Continuing in this way we find that for any $r = 1, \ldots, d$,

$$s(x) = \sum_{j=\mu-d+r}^{\mu} c_j^r B_{j,d-r}(x). \tag{6}$$

The case $r = d$ gives us $s(x) = c_\mu^d$. □

This algorithm can also be arranged in a triangular scheme, as follows. In each column, each value is computed from two values from the previous column.

$$\begin{array}{ccccc} c_{\mu-d}^0 & c_{\mu-d+1}^1 & \cdots & c_{\mu-1}^{d-1} & c_\mu^d \\ c_{\mu-d+1}^0 & c_{\mu-d+2}^1 & \cdots & c_\mu^{d-1} & \\ \vdots & & \iddots & & \\ c_{\mu-1}^0 & c_\mu^1 & & & \\ c_\mu^0 & & & & \end{array}$$

2 Marsden's identity

For each $j = 1, \ldots, n$, let us define the so-called dual polynomial

$$\rho_{j,d}(y) = (y - t_{j+1})(y - t_{j+2}) \cdots (y - t_{j+d}).$$

Then Marsden's identity is as follows.

Theorem 2 *For any $x \in [t_{d+1}, t_{n+1}]$ and for any $y \in \mathbb{R}$,*

$$(y - x)^d = \sum_{j=1}^{n} \rho_{j,d}(y) B_{j,d}(x). \tag{7}$$

To prove this theorem, it is sufficient to show a local form of the theorem.

Theorem 3 *If $x \in [t_\mu, t_{\mu+1})$, for some $\mu \in \{d+1, \dots, n\}$, then for any $y \in \mathbb{R}$,*

$$(y - x)^d = \sum_{j=\mu-d}^{\mu} \rho_{j,d}(y) B_{j,d}(x). \tag{8}$$

Proof. The proof uses Algorithm 2 applied to the initial data $c_j = \rho_{j,d}(y)$, $j = \mu - d, \dots, \mu$. Consider the first step of the algorithm. With $r = 1$ in (5),

$$\begin{aligned} c_j^1 &= \frac{t_{j+d} - x}{t_{j+d} - t_j} c_{j-1}^0 + \frac{x - t_j}{t_{j+d} - t_j} c_j^0 \\ &= \frac{t_{j+d} - x}{t_{j+d} - t_j} \rho_{j-1,d}(y) + \frac{x - t_j}{t_{j+d} - t_j} \rho_{j,d}(y) \\ &= \left(\frac{t_{j+d} - x}{t_{j+d} - t_j} (y - t_j) + \frac{x - t_j}{t_{j+d} - t_j} (y - t_{j+d}) \right) \rho_{j,d-1}(y), \end{aligned}$$

and a simple calculation shows that

$$\frac{t_{j+d} - x}{t_{j+d} - t_j} (y - t_j) + \frac{x - t_j}{t_{j+d} - t_j} (y - t_{j+d}) = y - x.$$

This shows that

$$c_j^1 = (y - x) \rho_{j,d-1}(y).$$

In the next step of the algorithm, with $r = 2$ in (5), we find, similarly, that

$$c_j^2 = (y - x)^2 \rho_{j,d-2}(y).$$

Continuing in this way, we find that for all $r = 1, \dots, d$,

$$c_j^r = (y - x)^r \rho_{j,d-r}(y). \tag{9}$$

The case $d = r$ gives $c_\mu^d = (y - x)^d$, which, by Theorem 1, proves (8). □

3 Linear independence of B-splines

We can use Marsden's identity to show that the B-splines $B_{1,d}, \ldots B_{n,d}$ are linearly independent with respect to the interval $[t_{d+1}, t_{n+1}]$. To this end, suppose that there are coefficients c_j such that

$$\sum_{j=1}^{n} c_j B_{j,d}(x) = 0, \qquad t_{d+1} \le x \le t_{n+1}. \tag{10}$$

The task is show that $c_1 = \cdots = c_n = 0$.

From (10), for any μ, $d+1 \le \mu \le n$,

$$\sum_{j=\mu-d}^{\mu} c_j B_{j,d}(x) = 0, \qquad t_\mu \le x \le t_{\mu+1}.$$

We will have $c_{\mu-d} = \cdots = c_\mu = 0$ if the B-splines $B_{\mu-d,d}, \ldots, B_{\mu,d}$ are linearly independent. Since there are $d+1$ of these, it is sufficient to show that we can express any monomial x^r, $0 \le r \le d$ as a linear combination of them. To do this we use the local form (8) of Marsden's identity. First we differentiate it $d-r$ times with respect to y, giving

$$\frac{d!}{r!}(y-x)^r = \sum_{j=\mu-d}^{\mu} \rho_{j,d}^{(d-r)}(y) B_{j,d}(x),$$

and then we let $y = 0$, giving

$$\frac{d!}{r!}(-1)^r x^r = \sum_{j=\mu-d}^{\mu} \rho_{j,d}^{(d-r)}(0) B_{j,d}(x).$$

Rearranging this gives

$$x^r = \sum_{j=\mu-d}^{\mu} c_{jr} B_{j,d}(x), \tag{11}$$

where

$$c_{jr} = (-1)^r \frac{r!}{d!} \rho_{j,d}^{(d-r)}(0).$$

Thus we have indeed now shown that $B_{\mu-d,d}, \ldots, B_{\mu,d}$ are linearly independent and so $c_{\mu-d} = \cdots = c_\mu = 0$. By considering all μ, it follows that $c_1 = \cdots = c_n = 0$, as claimed.

We can obtain an explicit formula for c_{jr} as follows. By the product rule for differentiation of a product of d functions, we have

$$\rho_{j,d}^{(d-r)}(y) = (d-r)! \sum_{j+1 \le j_1 < \cdots < j_r \le j+d} (y - t_{j_1}) \cdots (y - t_{j_r}).$$

The sum is over all possible products of r of the d factors of $\rho_{j,d}(y)$. It follows that

$$\rho_{j,d}^{(d-r)}(0) = (-1)^r (d-r)! \sum_{j+1 \le j_1 < \cdots < j_r \le j+d} t_{j_1} \cdots t_{j_r},$$

and therefore,

$$c_{jr} = \frac{1}{\binom{d}{r}} \sum_{j+1 \le j_1 < \cdots < j_r \le j+d} t_{j_1} \cdots t_{j_r}.$$

The sum here is over all possible products of r of the d interior knots $t_{j+1}, \ldots, t_{j+d}$ in the support of $B_{j,d}$. The binomial coefficient $\binom{d}{r}$ is the number of these products. Thus, c_{jr} is simply the average of all these products.

We can also express (11) as

$$x^r = \sum_{j=1}^{n} c_{jr} B_{j,d}(x). \tag{12}$$

Some examples are

$$\begin{aligned}
1 &= \sum_{j=1}^{n} B_{j,d}(x), \\
x &= \sum_{j=1}^{n} t_{j,d}^{*} B_{j,d}(x), \\
x^2 &= \sum_{j=1}^{n} t_{j,d}^{**} B_{j,d}(x), \\
x^d &= \sum_{j=1}^{n} t_{j+1} \cdots t_{j+d} B_{j,d}(x),
\end{aligned}$$

where

$$t_{j,d}^{*} = \frac{t_{j+1} + \cdots + t_{j+d}}{d},$$

$$t^{**}_{j,d} = \frac{t_{j+1}t_{j+2} + t_{j+1}t_{j+3} + \cdots + t_{j+d-1}t_{j+d}}{\binom{d}{2}}.$$

The first example shows that the B-splines sum to one at every x.