The Schoenberg-Whitney theorem and total positivity

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In these notes we use knot insertion to prove the Schoenberg-Whitney theorem and the total positivity of B-splines.

1 Refining B-splines by inserting a knot

Let us recall how a spline is refined by adding a new knot. Let $\mathbf{t}$ be a knot-vector, $\mathbf{t} = (t_1, t_2, \ldots, t_{n+d+1})$, and let $B_j = B_{j,d}$, $j = 1, 2, \ldots, n$, denote the j-th B-spline of degree d. Suppose that we add a new knot z to some interval $[t_\mu, t_{\mu+1})$, thus forming the refined knot vector

$$\tilde{\mathbf{t}} = (t_1, t_2, \ldots, t_\mu, z, t_{\mu+1}, \ldots, t_{n+d+1}).$$

Let $\tilde{B}_1, \ldots, \tilde{B}_{n+1}$ denote the B-splines on the knot vector $\tilde{\mathbf{t}}$. As we observed earlier,

$$B_j = \tilde{B}_j, \qquad j = 1, \ldots, \mu - d - 1,$$

and

$$B_j = \tilde{B}_{j+1}, \qquad j = \mu + 1, \ldots, n.$$

We further showed that for $j = \mu - d, \ldots, \mu$,

$$B_j = \begin{cases} \tilde{B}_j + \dfrac{t_{j+d+1} - z}{t_{j+d+1} - t_{j+1}} \tilde{B}_{j+1}, & j = \mu - d; \\ \dfrac{z - t_j}{t_{j+d} - t_j} \tilde{B}_j + \dfrac{t_{j+d+1} - z}{t_{j+d+1} - t_{j+1}} \tilde{B}_{j+1}, & \mu - d < j < \mu; \\ \dfrac{z - t_j}{t_{j+d} - t_j} \tilde{B}_j + \tilde{B}_{j+1}, & j = \mu. \end{cases}$$

It follows that for $j = 1, \ldots, n$,

$$B_j = \alpha_j \tilde{B}_j + \beta_j \tilde{B}_{j+1}, \tag{1}$$

where $\alpha_j, \beta_j \geq 0$ and

$$\begin{aligned} \alpha_j > 0 \quad & \text{if } t_j < z, \\ \beta_j > 0 \quad & \text{if } t_{j+d+1} > z. \end{aligned}$$

2 Schoenberg-Whitney theorem and total positivity

The Schoenberg-Whitney theorem provides a condition on a sequence of interpolation points that guarantees that the interpolation problem has a unique solution. We will show

Theorem 1 (Schoenberg-Whitney) *For an increasing sequence of points $x_1 < x_2 < \cdots < x_n$, the matrix*

$$A = [B_j(x_i)]_{i,j=1,\ldots,n}$$

is non-singular if and only if $B_i(x_i) > 0$ for all $i = 1, \ldots, n$.

Notice that if the multiplicity of all the knots is at most d then this condition is the same as the condition that $t_i < x_i < t_{i+d+1}$ for all i. However, we often choose the knot vector to have the first $d+1$ knots equal, and the last $d+1$ knots equal. Then the condition $B_1(x_1) > 0$ is equivalent to $t_1 \leq x_1 < t_{d+2}$, and the condition $B_n(x_n) > 0$ is equivalent to $t_n < x_n \leq t_{n+d+1}$. This allows the case that $x_1 = t_1$ and $x_n = t_{d+n+1}$. Therefore, the theorem can be applied to show that, for example, C^2 cubic spline interpolation with free end conditions is uniquely solvable.

In fact, will will also show that the matrix A is totally positive by which we mean that every minor of A is non-negative. Thus, we can prove both the Schoenberg-Whitney theorem and total positivity by proving the following more general theorem:

Theorem 2 *For any increasing sequence of points $x_1 < x_2 < \cdots < x_m$, where $m \leq n$, and any increasing sequence $\mathbf{j} = (j_1, j_2, \ldots, j_m)$ with $1 \leq j_1 < j_m \leq n$, the matrix*

$$A(\mathbf{j}) = [B_{j_k}(x_i)]_{i,k=1,\ldots,m}$$

has non-negative determinant. The determinant is positive if and only if $B_{j_i}(x_i) > 0$ *for all* $i = 1, \ldots, m$.

Proof. Suppose first that for some p, $B_{j_p}(x_p) = 0$. Then there are two possible cases, either $x_p \le t_{j_p}$ or $x_p \ge t_{j_p+d+1}$. If $x_p \le t_{j_p}$ then $B_{j_k}(x_i) = 0$ for all $i \le p$ and $k \ge p$. Then the first p rows of $A(\mathbf{j})$ are linearly dependent and $A(\mathbf{j})$ is singular. If on the other hand $x_p \ge t_{j_p+d+1}$ then $B_{j_k}(x_i) = 0$ for all $i \ge p$ and $k \le p$. The first p columns of $A(\mathbf{j})$ are linearly dependent and $A(\mathbf{j})$ is again singular.

It remains to consider the case that $B_{j_i}(x_i) > 0$ for all $i = 1, \ldots, m$. Suppose that there are at least $d+1$ knots in $\mathbf{t}$ between each consecutive pair of interpolation points x_i and x_{i+1}. Since x_i belongs to $[t_{j_i}, t_{j_i+d+1}]$, any other point x_p, $p \ne i$, cannot belong to $[t_{j_i}, t_{j_i+d+1}]$. Thus $B_{j_i}(x_p) = 0$ if $p \ne i$, and $A(\mathbf{j})$ is a diagonal matrix and since its diagonal elements are positive, it is non-singular with positive determinant.

Otherwise, we use induction on the number of knots between pairs of points x_p and x_{p+1}. Suppose that between some pair of points x_p and x_{p+1}, there are less than $d+1$ knots of $\mathbf{t}$. We now form a new knot vector $\tilde{\mathbf{t}}$ by adding a new knot z to $\mathbf{t}$ between x_p and x_{p+1}. Then, recalling (1), and by the linearity of the determinant of $A(\mathbf{j})$ with respect to its columns,

$$\det A(\mathbf{j}) = \sum_{\boldsymbol{\epsilon} \in \{0,1\}^m} \gamma_{\boldsymbol{\epsilon}} \det \tilde{A}(\mathbf{j} + \boldsymbol{\epsilon}), \tag{2}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$, and

$$\gamma_{\boldsymbol{\epsilon}} = \prod_{k=1}^{m} \big((1 - \epsilon_k)\alpha_{j_k} + \epsilon_k \beta_{j_k}\big) \ge 0,$$

and

$$\tilde{A}(\mathbf{j} + \boldsymbol{\epsilon}) = [\tilde{B}_{j_k + \epsilon_k}(x_i)]_{i,k=1,\ldots,m}.$$

Any $\mathbf{j} + \boldsymbol{\epsilon}$ in (2) is a non-decreasing subsequence of $(1, 2, \ldots, n+1)$. If any two consecutive elements of $\mathbf{j} + \boldsymbol{\epsilon}$ are equal then two of the columns of $\tilde{A}(\mathbf{j} + \boldsymbol{\epsilon})$ are equal and $\det \tilde{A}(\mathbf{j} + \boldsymbol{\epsilon}) = 0$. Therefore, we can remove such $\boldsymbol{\epsilon}$ from the sum in (2), and we have

$$\det A(\mathbf{j}) = \sum_{\substack{\boldsymbol{\epsilon} \in \{0,1\}^m \\ \mathbf{j}+\boldsymbol{\epsilon} \text{ increasing}}} \gamma_{\boldsymbol{\epsilon}} \det \tilde{A}(\mathbf{j} + \boldsymbol{\epsilon}). \tag{3}$$

By the induction hypothesis, all the determinants in the sum in (3) are non-negative, and this implies, by induction, that $\det A(\mathbf{j}) \geq 0$. To show that $\det A(\mathbf{j}) > 0$ we must show that there is at least one $\boldsymbol{\epsilon} \in \{0,1\}^m$ for which $\mathbf{j} + \boldsymbol{\epsilon}$ is increasing, $\gamma_{\boldsymbol{\epsilon}} > 0$, and $\det \tilde{A}(\mathbf{j} + \boldsymbol{\epsilon}) > 0$. Indeed this is true for $\boldsymbol{\epsilon}^* = (\epsilon_1^*, \ldots, \epsilon_m^*)$, where

$$\epsilon_1^* = \cdots = \epsilon_p^* = 0, \qquad \epsilon_{p+1}^* = \cdots = \epsilon_m^* = 1.$$

To see this observe that $\mathbf{j} + \boldsymbol{\epsilon}^*$ is clearly increasing. Next, suppose $1 \leq k \leq p$. Then, since $t_{j_k} \leq x_k \leq t_{j_k+d+1}$, it follows that $t_{j_k} \leq x_k \leq x_p < z$ and so $\alpha_{j_k} > 0$. Also, since $B_{j_k}(x_k) > 0$ and since $x_k < z$, it follows that $\tilde{B}_{j_k}(x_k) > 0$. The other case is that $p + 1 \leq k \leq m$. Then, since $t_{j_k} \leq x_k \leq t_{j_k+d+1}$, it follows that $t_{j_k+d+1} \geq x_k \geq x_{p+1} > z$ and so $\beta_{j_k} > 0$. Also, since $B_{j_k}(x_k) > 0$ and since $x_k > z$, it follows that $\tilde{B}_{j_k+1}(x_k) > 0$. Therefore, $\gamma_{\boldsymbol{\epsilon}^*} > 0$ and, by the induction hypothesis, $\det \tilde{A}(\mathbf{j} + \boldsymbol{\epsilon}^*) > 0$, as claimed. $\square$